Topological Combinatorics

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Basic Topology

Definitions

Topology is study of geometric objects under continuous deformations transformation that do not cut the object.

Definition

X is some set and τ is a set of subsets of X. (X, τ) is topological space if

1. The empty set and X are in τ
2. The union of any number elements of τ is in τ
3. The intersection of any number elements of τ is in τ

Additionally, for any topological space (X, τ), τ is said to be a *topology* on X.

Theorems

Main theorem we will be using:

Theorem (Borsuk-Ulam)

If the function $f : S^n \to \mathbb{R}^n$ is continuous, there exists $x \in S^n$ such that $f(x) = f(-x)$.

The Necklace Problem

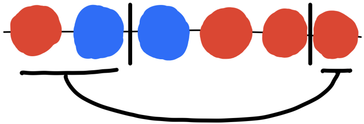
The Problem

Proposition

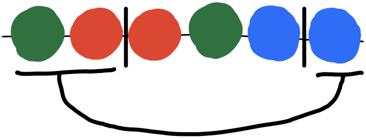
Let there be $2d > 2$ jewels on a string where each jewel is one of n types. It is always possible to use n or fewer cuts to cut and divide the substrings among two people where each person gets the same number of jewels of each type.

Examples

Example



Example



Only required two cuts.

Proof

- Convert problem to continuous one: Jewels $\rightarrow$ regions in $[0, 1]$
- $$S^n = \{x \in R^{n+1} \mid x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}.$$
  Cutting with n cuts $\rightarrow$ finding $x_1^2, x_2^2, \cdots, x_{n+1}^2$.
  Sign $\rightarrow$ which person gets the piece.
- $f : S^n \rightarrow \mathbb{R}^n$ equals function that maps cutting division to $\mathbb{R}^n$. Antipodal points mean switching who gets what piece.
- $f(x) = f(-x) \rightarrow$ switching doesn't matter $\rightarrow$ fair division.

The Inscribed Rectangle Problem

Problem

Fairly famous unsolved problem is the inscribed square problem:

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Can we always inscribe a square in any loop?

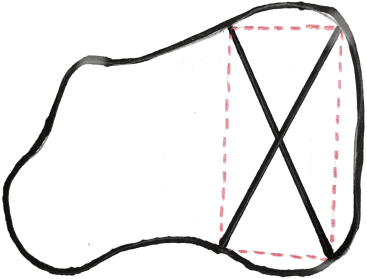
Weaker version of this problem that we can prove is the inscribed rectangle problem:

Proposition

It is always possible to inscribe a rectangle in any loop.

Proof

- Rectangle $\rightarrow$ two line segments of equal length that share midpoint



Proof (cont.)

- Put loop on xy-plane. Above midpoint of each line segment, z coord $=$ length of line segment. Now we get surface.
- Loop $\rightarrow$ line segment. x and y axes $=$ the line segment. Glue to get Möbius Strip.
- Map Möbius Strip onto surface. Edge lines up with loop and surface intersects itself.

Thank you

Thank you for your attention!