Rotational Motion

Fun With Fiziks

Rotational Kinematics

- Rotational kinematics is simply kinematics but in circles.
- All the kinematics equations are the same but in rotational variables.
- The new unit, radians, is introduced when the radius of circular motion is involved.
- **Velocity v is now ω** with new units of rad/s instead of m/s
- **Displacement x is now θ** with new units of rad instead of m.
- **Acceleration a is now α** with rad/s2 instead of m/s2

1. $\omega = \omega_0 + \alpha t$
2. $\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$
3. $\omega^2 = \omega_0^2 + 2\alpha(\theta - \theta_0)$
4. $\theta - \theta_0 = \frac{1}{2}(\omega_0 + \omega)t$

Rotational Kinematics Summary

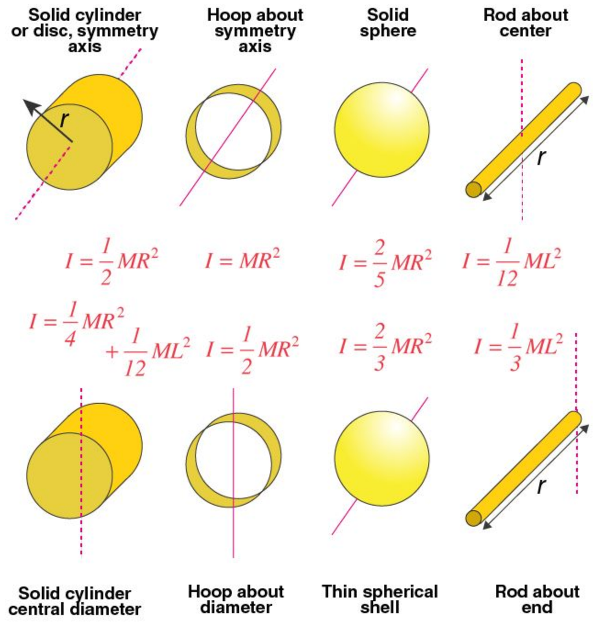
- Conversion between linear and rotational motion:

Linear	Type	Rota-tional	Relation (θ in radians)
x	displacement	θ	$x = R\theta$
v	velocity	ω	$v = R\omega$
a_{tan}	acceleration	α	$a_{tan} = R\alpha$

Rotational Motion (α = constant)	Linear Motion (a = constant)
$\omega = \omega_0 + \alpha t$	$v = v_0 + at$
$\theta = \frac{1}{2}(\omega_0 + \omega)t$	$x = \frac{1}{2}(v_0 + v)t$
$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	$x = v_0 t + \frac{1}{2}at^2$
$\omega^2 = \omega_0^2 + 2\alpha\theta$	$v^2 = v_0^2 + 2ax$

What is the moment of inertia?

- Rotational version of mass
- How hard something is to spin
  - Normal mass: more mass → harder to move
  - Moment of inertia: more rotational inertia → harder to rotate
- Different from normal mass because it depends on the **shape of the object** and the **axis of rotation**
- Moment of inertia of a point particle = MR^2



Moment of inertia cont.

- The moment of inertia depends on the radius
  - If R increases, then rotational inertia increases
- This tells us that the further the mass is from the axis of rotation, the harder something is to spin

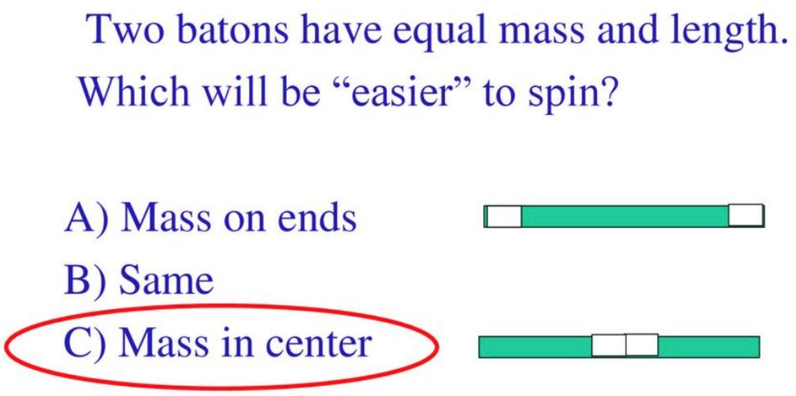
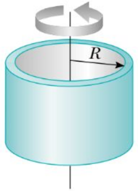
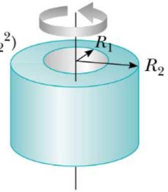


TABLE 10.2 Moments of Inertia of Homogeneous Rigid Bodies with Different Geometries

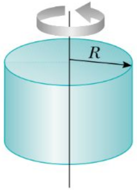
Hoop or cylindrical shell
$I_{CM} = MR^2$



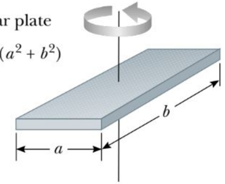
Hollow cylinder
$I_{CM} = \frac{1}{2} M(R_1^2 + R_2^2)$



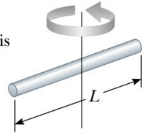
Solid cylinder or disk
$I_{CM} = \frac{1}{2} MR^2$



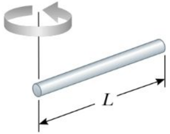
Rectangular plate
$I_{CM} = \frac{1}{12} M(a^2 + b^2)$



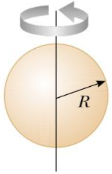
Long thin rod with rotation axis through center
$I_{CM} = \frac{1}{12} ML^2$



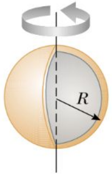
Long thin rod with rotation axis through end
$I = \frac{1}{3} ML^2$



Solid sphere
$I_{CM} = \frac{2}{5} MR^2$



Thin spherical shell
$I_{CM} = \frac{2}{3} MR^2$



Rotational Energy

KE = ½ mv^2

- If we consider a rotating rigid object as a collection of particles, each particle has the aforementioned kinetic energy
- Every particle has the same ω, and it's v depends on the distance from the axis:

$$K_R = \sum_i K_i = \sum_i \tfrac{1}{2} m_i v_i^{\,2} = \tfrac{1}{2} \sum_i m_i r_i^{\,2} \omega^2 \qquad K_R = \tfrac{1}{2}\left(\sum_i m_i r_i^{\,2}\right)\omega^2$$

- We can define the quantity in parentheses as the moment of inertia I:

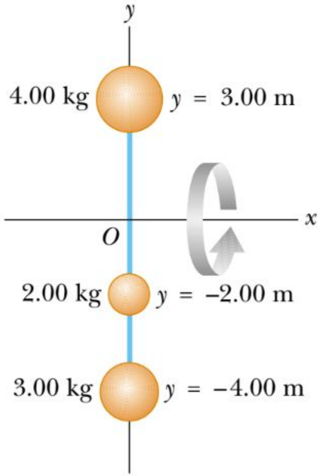
$$K_R = \tfrac{1}{2} I \omega^2$$

$$I \equiv \sum_i m_i r_i^{\,2}$$

Insert easy rotational energy problem

Ex: a wheel (hollow cylindrical shell) of mass 3 kg and radius 1 m is spinning with an angular velocity of 5 rad/s. What is its rotational kinetic energy?

23. Three small particles are connected by rigid rods of negligible mass lying along the y axis (Fig. P10.23). If the system rotates about the x axis with an angular speed of 2.00 rad/s, find (a) the moment of inertia about the x axis and the total rotational kinetic energy evaluated from $\frac{1}{2}I\omega^2$ and (b) the linear speed of each particle and the total kinetic energy evaluated from $\Sigma\frac{1}{2}m_i v_i^2$.



Hint: moment of inertia of a point particle = MR^2

a)

$I = \sum m_i r_i^2$

$4(3)^2 + 2(-2)^2 + 3(-4)^2$

$36 + 8 + 48 = \boxed{92\ kg \cdot m^2}$

$K_r = \frac{1}{2} I w^2$

$\frac{1}{2}(98)\,2^2 = 92 \cdot 2 = \boxed{184\ J}$

b)

$2 \cdot 3 = 6\ m/s$

$2 \cdot 2 = 4\ m/s$

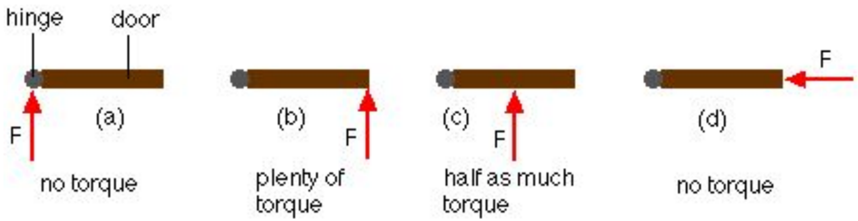
$2 \cdot 4 = 8\ m/s$

$\sum \frac{1}{2} m_i v_i^2$

$\frac{1}{2}\left(4 \cdot 6^2 + 2 \cdot (-4)^2 + 3 \cdot (-8)^2\right)$

$\boxed{184\ J}$

What is torque, exactly?



- Torque (τ) is NOT a force, although it’s convenient to think of it as the “turning force”
  - It is a “measure of the force that can cause an object to rotate about an axis”, or obtain angular acceleration
  - Other words for torque: “moment”, “moment of force”
- A commonly used explanation: if you push on a door harder or farther away from its hinges, you’re more likely to open it
  - If you push on a door close to its hinges, you’ll need a greater force to open it than if you push away from its hinges.
  - Both movements do the same work (the door has the same displacement)
- Torque can be static or dynamic:
  - Static torque doesn’t produce angular acceleration: (pushing on a closed door, pedaling a bicycle at constant speed)

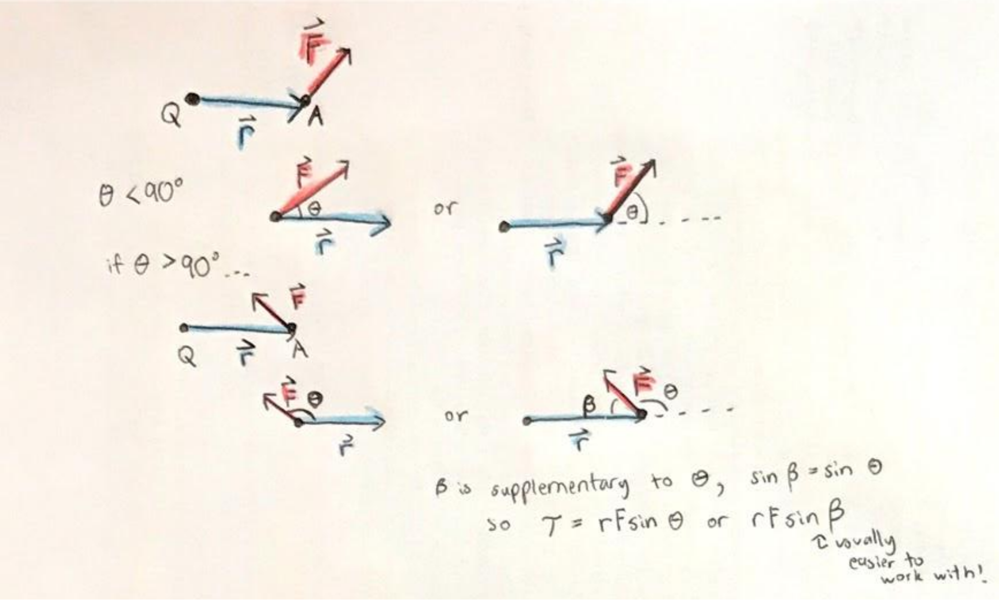
$$\vec{\tau} = \vec{r} \times \vec{F}$$

- Torque is the cross product between the position vector and the force vector
- *r* is the distance between the **pivot point** and the point of application of **F**
  - Has many names: lever arm, “moment”, “moment arm”
- Units of torque: Newton meters

Magnitude of torque

$$|\vec{\tau}| = rF\sin\theta$$

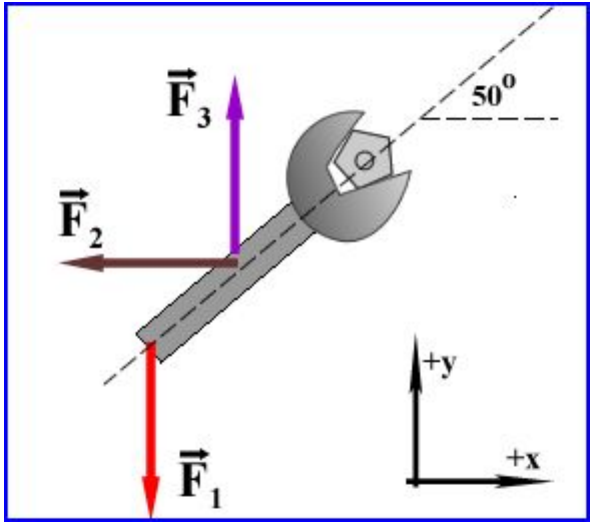
- The angle θ is the angle between the position vector and the force vector
- So to calculate the magnitude of the torque, take the sine of the smallest angle between the force and the line along the position vector



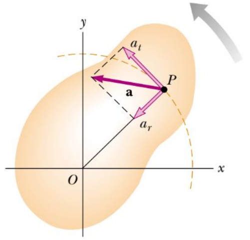
Insert easy practice problem (literally plug into formula)

Ex: There is a force of 10 N on a wrench 3 m from the point of rotation. What is the torque?

In the figure, a wrench is used to tighten or loosen a bolt. Calculate the torque about an axis passing through the center of the bolt due to each of the forces shown in the figure. The distance between the center of the bolt and the end of the wrench is **2d**. Force $\mathbf{F_1}$ is applied at the end of the wrench, and forces $\mathbf{F_2}$ and $\mathbf{F_3}$ are applied at a distance **d** from the center of the bolt. Express your answer in terms of the magnitude of the forces, the distance d and the appropriate trigonometric functions.



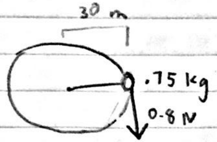
Torque and Angular Acceleration



- A particle of mass *m* rotating in a circle with radius *r* under $\mathbf{F_t}$ and $\mathbf{F_r}$
  - $F_t = ma_t$ — Torque about center of the circle due to $\mathbf{F_t} = \tau = F_t r = (ma_t)r$
  - $\tau = (mr\alpha)r = mr^2\alpha$ — mr^2 is the moment of inertia of a particle
  - $\tau = I\alpha$ — note similarity between Newton's 2nd law?
- What about an object?
  - Basically a collection of infinite number of mass elements *dm* of infinitesimal size (each with different $\mathbf{a}_t$ but same $\boldsymbol{\alpha}$)
  - $dF_t = (dm)a_t$ — Small change in torque about center of the circle due to $\mathbf{F_t} = d\tau = dF_t{*}r = (r{*}dm)a_t$
  - $d\tau = (r{*}dm)r\alpha = (r^2\,dm)\alpha$ — α is constant: $\Sigma\tau = \int(r^2\,dm)\alpha = \alpha\int(r^2\,dm)$
  - $\Sigma\tau = I\alpha$

37. A model airplane having a mass of 0.750 kg is tethered by a wire so that it flies in a circle 30.0 m in radius. The airplane engine provides a net thrust of 0.800 N perpendicular to the tethering wire. (a) Find the torque the net thrust produces about the center of the circle. (b) Find the angular acceleration of the airplane when it is in level flight. (c) Find the linear acceleration of the airplane tangent to its flight path.

37



a) $\tau = F r \sin\theta$

$24 = 0.8 \cdot 30 (1)$

$\boxed{24 \text{ N.m}}$

b) $\tau = I\alpha$

treat plane as particle

$I = MR^2$

$(0.75)(30)^2 = 675 \text{ kg m}^2$

$24 = 675 \cdot \alpha$

$\boxed{\alpha = 0.0356 \text{ rad/s}^2}$

c) $a_t = r\alpha$

$30(0.0356) = \boxed{1.068 \text{ m/s}^2}$

Ball rolling down an incline → explain the lab

What is the acceleration of a rolling object down an incline?

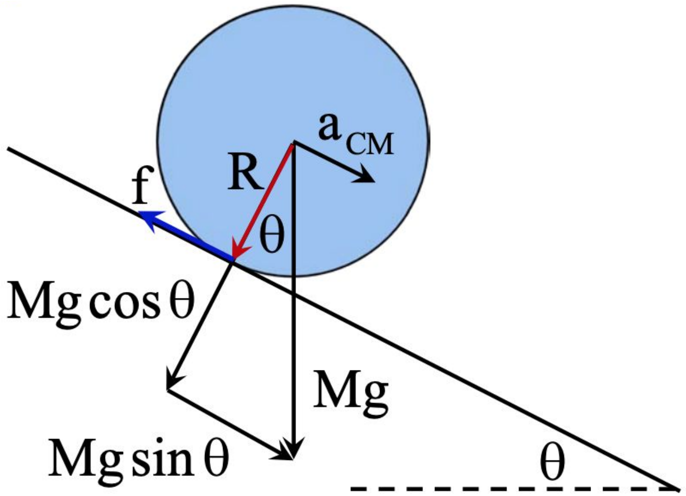
Use forces and torque - newton's 2nd law

$$Mg\sin\theta - f = Ma_{CM}$$

$$\tau = Rf = I\alpha$$

2 equations with 2 unknowns: a_{CM} and f

Solve for a_{CM}



Ball rolling down an incline cont

Notice that if I is bigger, acceleration is smaller!

Things harder to rotate accelerate slower down the incline

$$f = \frac{I}{R}\alpha \rightarrow Mg\sin\theta - \frac{I}{R}\alpha = Ma_{CM}$$

$$\alpha = \frac{a_{CM}}{R} \rightarrow Mg\sin\theta - \frac{I}{R}\frac{a_{CM}}{R} = Ma_{CM}$$

$$Ma_{CM} + \frac{I}{R^2}a_{CM} = Mg\sin\theta$$

$$Ma_{CM}\left(1 + \frac{I}{MR^2}\right) = Mg\sin\theta$$

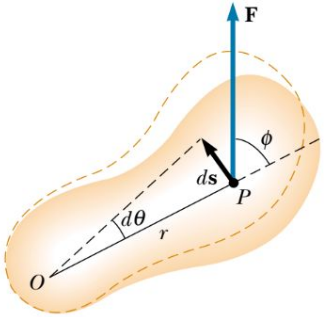
$$a_{CM}\left(1 + \frac{I}{MR^2}\right) = g\sin\theta$$

$$\boxed{a_{CM} = \frac{g\sin\theta}{1 + \frac{I}{MR^2}}}$$

Ex: if a solid cylinder was rolling down the incline, plug in I = 1/2MR^2 into the equation

$$a_{CM} = \frac{g\sin\theta}{1 + \frac{\frac{1}{2}MR^2}{MR^2}} = \frac{g\sin\theta}{1 + \frac{1}{2}} = \frac{g\sin\theta}{\frac{3}{2}} = \boxed{\frac{2}{3}g\sin\theta}$$

Work, power, and energy



- $W = \mathbf{F} \cdot \mathbf{d} = Fd\cos\theta$
    - $d = r\theta$
    - $W = \mathbf{F}d = (Fr)\theta = \tau\theta$
- Power is the rate at which work is done. (divide by t)

$$\frac{dW}{dt} = \tau \frac{d\theta}{dt} \qquad \mathcal{P} = \frac{dW}{dt} = \tau\omega$$

- The work-kinetic energy theorem also applies to rotational motion:
    - $W = \Delta K_{rot} = ½ * I * \omega_f\text{^}2 - ½ * I * \omega_i\text{^}2$

Angular momentum

L = I * omega = mv * r

Similar form to p = mv since m → I and v → omega for rotations

Angular momentum is conserved when there is no external torque (just like linear momentum except no external force)

I * omega is used for rotating things

mv * r is used when something moving linearly transfers its momentum into something that rotates

Spinny chair demo

When you hold out your arms you are increasing moment of inertia so your angular velocity decreases since angular momentum is conserved

Practice problem

A figure skater is holding their arms out 0.5 m from their body and is spinning at 3 rad/s. Then, they tuck their arms in 0.25 m from their body. How fast do they spin?

$I_i\omega_i = I_f\omega_f$

$cmr_i^2 * \omega_i = cmr_f^2 * \omega_f$ (c is just any constant because we technically don't know the shape of the figure skater → turns out this doesn't matter!)

$r_i^2 * \omega_i = r_f^2 * \omega_f$

$\omega_f = r_i^2 * \omega_i / r_f^2 = 0.5^2 * 3/0.25^2 =$ 12 rad/s