Problem Set Solutions for

Real Analysis

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Contents

1 Week 1

(1) The Euclidean distance in $\mathbb{R}$ between elements x and y is $|x-y|$. Now we check if this expression satisfies all the metric properties. The first property is an if and only if statement so we first prove the forward direction: if $d(x,y) = |x-y| = 0$, then $x = y$. We see $|x-y| = 0 \implies x-y = 0 \implies x = y$. For the other direction, if $x = y$, we have $|x-y| = |x-x| = 0$. Next, for the second property we have

$$d(x,y) = |x-y| = |-(x-y)| = |y-x| = d(y,x).$$

For the last property, we introduce a new element $z \in \mathbb{R}$. WLOG we assume $x \geq y \geq z$. Now the triangle inequality states

$$|x-y| \leq |x-z| + |z-y|.$$

Using our assumption, we get

$$x - y \leq x - z + y - z = x + y - 2z.$$

Simplifying this further gives us $y \leq z$ which we know is true. Since all our steps are reversible we are done proving the Triangle Inequality. Therefore we are done proving that standard Euclidean distance in $\mathbb{R}$ is in fact a metric.

Now we follow similar steps for Euclidean distance in $\mathbb{R}^2$. We know the distance between points (a,b) and (c,d) is $\sqrt{(a-c)^2 + (b-d)^2}$. For the first property, we can see that if the points are the same, the distance is 0. For the reverse, we have

$$\sqrt{(a-c)^2 + (b-d)^2} = 0 \implies (a-c)^2 + (b-d)^2 = 0 \implies (a,b) = (c,d).$$

Property 2 follows from the fact that squaring results is a positive. Finally, for the last property, if the third point is (e,f), we have

$$\sqrt{(a-c)^2 + (b-d)^2} \leq \sqrt{(a-e)^2 + (b-f)^2} + \sqrt{(e-c)^2 + (f-d)^2}.$$

After going through the arithmetic and since all our steps are reversible, we can see that this is true. Therefore, Euclidean distance in $\mathbb{R}^2$ is a metric.

(2)

(a) Property 1 holds by definition. Property 2 is also trivial. For the triangle inequality property we first go with the case that $x = y$. This means that $d(x,x) = 0$ and $d(x,z) + d(z,y) = 2d(x,z)$ which is greater than or equal to 0. When $x \neq y$, we have $d(x,y) = 1$. Now for the other side of the inequality, we see that it must always be greater than 0 since x and y cannot equal z at the same time.

(b) When $x = y$, we know that the max function is either 0 or 1 but $d(x,y) = 1$ so the property is satisfied. When $x \neq y$, we the max function is 1 while $d(x,y) = 0$. Therefore our metric is an ultrametric.

(3) If $d_p(x,y) = 0$ then by we know, by definition, that the only way this can happen is if $x - y = 0 \implies x = y$. For the reverse, we have $|x-y|_p = |0|_p = 0$. Now we go on to the second property. If we let $x - y = mp^n$ where $m \neq 0$ and is relatively prime to p, we have

$$|y-x|_p = |-(x-y)|_p = |-mp^n|_p = p^{-n} = |x-y|_p.$$

The reason $|-mp^n|_p = p^{-n}$ is because $-m$ is still not 0 and not divisible by p.

The last property we have to prove is the Triangle Inequality. Let $x = kp^a$ and $y = lp^b$ where rational k and l are not 0 and $|k|_p = |l|_p = 1$. WLOG assume that $a \leq b$. We must prove that

$$|x + y|_p \leq |x|_p + |y|_p.$$

The LHS is just $|kp^a + lp^b|_p = p^{-a}|k + lp^{b-a}|_p$. If $b \neq a$, since p does not divide $(k + lp^{b-a})$ we have

$$|x + y|_p = p^{-a}|k + lp^{b-a}|_p = p^{-a} \leq p^{-a} + p^{-b} = |x|_p + |y|_p.$$

Now if $a = b$, the LHS is $p^{-a}|k + lp^{b-a}|_p = p^{-a}|k + l|_p$. If $k + l = p^r$, we have

$$|x + y|_p = p^{-a}|k + l|_p = p^{-a-r} \leq 2p^{-a} = |x|_p + |y|_p.$$

Finally, if $p \nmid (k + l)$, we have

$$|x + y|_p = p^{-a}|k + l|_p = p^{-a} \leq 2p^{-a} = |x|_p + |y|_p.$$

(4)

(a) For $\varepsilon > 0$, let N be an integer such that $N^{-1} < \frac{\varepsilon}{2}$ and let $n, m > N$. We see that

$$d(a_n, a_m) = \left|\frac{1}{n} - \frac{1}{m}\right| \leq \left|\frac{1}{n}\right| + \left|-\frac{1}{m}\right| = \frac{1}{n} + \frac{1}{m} < \frac{2}{N} < \varepsilon.$$

(b) Again for $\varepsilon > 0$, let N be an integer such that $N^{-1} < \frac{\varepsilon}{2}$ and let $n, m > N$. We see that

$$d(b_n, b_m) = \left|\frac{\sin n}{n} - \frac{\sin m}{m}\right| \leq \left|\frac{1}{n} + \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} \leq \frac{2}{N} < \varepsilon$$

(c) Since $1/n$ and $1/n^2$ are both Cauchy sequences, this combination must also be one.

(d) As the sequence goes on, the terms are getting closer at at least a rate of 8×10^{-n}. Since we already proved that $(a_n) = 10^{-n}$ is a Cauchy sequence, we are done.

(5) Let $\varepsilon > 0$ be some arbitrary number. Let N be an integer such that when $n, m > N$, we have $|x_n - x_m| < \varepsilon/2$ and let M be an integer such that when $n, m > M$, we have $|y_n - y_m| < \varepsilon/2$. Finally, let $N' = \max(N, M)$. Therefore when $n, m > N'$ we have

$$|x_n + y_n - (x_m + y_m)| = |x_n - x_m + y_n - y_m| \leq |x_n - x_m| + |y_n - y_m| < \varepsilon$$

(6) We can go through similar steps like the previous problem to get

$$|x_n y_n - x_m y_m| = |x_n(y_n - y_m) + y_m(x_n - x_m)| < |x_n(y_n - y_m)| + |y_m(x_n - x_m)|.$$

Since (x_n) and (y_m) are Cauchy, they are bounded so we have

$$|x_n(y_n - y_m)| + |y_m(x_n - x_m)| < |x_n - x_m| + |y_n - y_m| < \varepsilon.$$

(7)

(a) For the sake of contradiction assume (a_n) is Cauchy. This means that for any $\varepsilon > 0$, there exists a N such that $|a_n - a_m| = |n - m|$ whenever n and m are integers greater than N. However we can see that this is not true for $\varepsilon = 0.5$.

(b) For the sake of contradiction assume (b_n) is Cauchy. This means that for any $\varepsilon > 0$, there exists a N such that $|b_n - b_m| = |\sqrt{n} - \sqrt{m}|$ whenever n and m are integers greater than N. Let $n = 4m$. Notice that n, m are still greater than N. Now we have $|\sqrt{4m} - \sqrt{m}| = \sqrt{m}$ which can be greater than ε so we have a contradiction.

(c) The difference between terms can be 0 or ± 2 so the sequence cannot be Cauchy.

2 Week 2

(1) For $(x_n + y_n)$, let $\varepsilon > 0$ be arbitrary. Let N be an integer where $n > N \implies |x_n| < \varepsilon/2$ and similarly, let M be an integer where $n > M \implies |y_n| < \varepsilon/2$. Now let $N' = \max(N, M)$. Now for $n > N'$, we have

$$|x_n + y_n| < |x_n| + |y_n| < \varepsilon.$$

For $(x_n \cdot y_n)$, let N be an integer where $n > N \implies |x_n| < \sqrt{\varepsilon}$ and similarly, let M be an integer where $n > M \implies |y_n| < \sqrt{\varepsilon}$. Now let $N' = \max(N, M)$. When $n > N'$, we have

$$|x_n \cdot y_n| = |x_n| \cdot |y_n| < \sqrt{\varepsilon} \cdot \sqrt{\varepsilon} = \varepsilon$$

(2)

1. Let $\varepsilon > 0$ be arbitrary. Since (x_n) is Cauchy, we have $|x_n - x_m| < \varepsilon/2$ when $n, m > N$ for some integer N and since $y_n \to 0$, we have $|y_n| < \varepsilon/2$ when $n > M$ for some integer M. Let $N' = \max(N, M)$. When $n, m > N'$, we have

$$|x_n y_n| = |y_n(x_n - x_m) + x_m y_n| < |x_n - x_m| + |y_n| < \varepsilon$$

2. Let $\varepsilon > 0$ be arbitrary. If $x_n \to 0$, for an integer N, we know that $n > N \implies |x_n| = ||x_n|| < \varepsilon$ which means that $|x_n|$ must converge to 0. If $|x_n| \to 0$, we know that when $n > N$, we have

$$||x_n|| = |x_n| < \varepsilon$$

which is the definition of $x_n \to 0$.

(3) Let $\varepsilon > 0$ be arbitrary. Since (x_n) is Cauchy, for some integer N, we know that $n, m > N \implies |x_n - x_m| < \varepsilon$. Now we note that $k_n > n$. This means that $n, k_n > N$ so since (x_n) is Cauchy,

$$|x_n - x_{k_n}| < \varepsilon.$$

(4) The set of all subsequences includes the subsequence that starts from the second term and goes on and we know that this subsequence converges to x. Therefore adding one term to the beginning to create (x_n) will not change the convergence.

(7)

(a) We can see that this metric is symmetric since d_1 and d_2 are both symmetric. Now if $x = y$, we know that $d_1(x, y) = 0$ and $d_2(x, y) = 0$ so $d_1 + d_2 = 0$. If $d_1 + d_2 = 0 \implies d_1 = 0, d_2 = 0$. Since $d_1 = 0, d_2 = 0 \implies x = y$. Now for triangle inequality we can just add the triangle inequalitys for d_1 and d_2.

(9)

(a) If x is positive, we have $x < \varepsilon$ and if x is negative, we have $x > -\varepsilon$. For the reverse, if $-\varepsilon < x < \varepsilon$, we know that x must be at most a distance of ε away from zero so $|x| < \varepsilon$.

(b) We know through triangle inequality that

$$|y| + |x - y| \geq |x|$$

and

$$|x| + |y - x| \geq |y|.$$

Therefore
$$|x - y| \geq |x| - |y|$$
and
$$|y - x| = |x - y| \geq |y| - |x| = -(|x| - |y|).$$
We can see that these two inequalities imply that $|x - y| \geq ||x| - |y||$.

(11) For the sake of contradiction, let $x > y$. Using this fact and also that $x_n < y$ for all n, we see that $|x_n - x| > x - y$. Therefore when some real number ε is in the range $(0, x - y]$,
$$|x_n - x| > \varepsilon$$
which contradicts (x_n) converging to x.

3 Week 3

(1) The least upperbound in $\mathbb{R}$ is $\sqrt{2}$. Therefore the least upperbound in $\mathbb{Q}$ is the number greater than $\sqrt{2}$ and as close as possible to it. However if we think some rational number r is the closest to $\sqrt{2}$, we can always pick a rational r' such that $\sqrt{2} < r < r$ so the sequence does not have a least upperbound.

(2) For $\mathbb{N}, \mathbb{Z}$, they are not fields which breaks axiom 1. For $\mathbb{C}$, this set does not follow the ordering axiom since there is no well defined way of ordering complex numbers.

(4)

(a) Let (x_n) be a representative of x. Since (x_n) is Cauchy, it is bounded so there must be a rationa number r such that $x_n \geq r$ for all n. Therefore, all we need to do now is to pick a positive integer m such that $m > 1/r$.

(b) Since $x < y \implies 0 < y - x$, there exists an integer n such that
$$0 < \frac{1}{n} < y - x$$
from part (a). Now, we also know that every integer n can be written as $m/2$ for even m so
$$\frac{1}{n} < y - x \implies \frac{2}{m} < y - x \implies x + \frac{1}{m} < y - \frac{1}{m}.$$

(c) If $x = 0$, then $x = [(0)]$ and $|x| = |[(0)]| = [(|0|)] = [(0)]$. This is the equivalence class that convergs to 0. This means that its terms become arbitrarily close to 0. Therefore, by definition of convergence, there will always be an integer. n such that $n > N \implies x_n < \varepsilon$ which means that $|x| < \varepsilon$.

For the converse, we know that x is the equivalence class such that there exists an N such that $n > N \implies |x_n| < \varepsilon$. Therefore $(|x_n)|) \to 0$ so $|x| = 0 \implies x = 0$.

(5)

(a) For a set S, the greatest lower bound x is a number such that $x \leq s$ for all $s \in S$. and there exists no $x' > x$ where $x' \leq s$.

(b) We know that $x < b$ for all $x \in S$ so $-x > -b$, through basic properties of inequalities. So $-b$ is the greatest lower bound.

(8) We must find a sequence that becomes arbitrarily big and becomes arbitrarily small. The following recursively defined sequence does this:

$$a_n = \begin{cases} -a_{n-1} + 1 & n \equiv 1 \pmod 2 \\ -a_{n-1} & n \equiv 0 \pmod 2 \end{cases}$$

where $a_0 = 0$. We can notice that this sequence will have no lower or upper bound.

(9) We know that the lim inf $= -1$ and the lim sup $= 1$. Since these values are not equal, this sequence does not converge.

4 Week 4

(1)

(a) We claim that $\lim_{x\to\infty} 1/x = 0$. For an arbitrary ε, if we let $M > 1/\varepsilon$, we know that for all $m > M$,

$$\left|\frac{1}{m} - 0\right| < \frac{1}{M} < \varepsilon$$

so the limit is 0. However the function $f(x) = x$ does not exists when $x \to \infty$ since this function gets arbitrarily big as x goes to infinity.

(3)

(a) Fix arbitrary y, let $\varepsilon > 0$ and let $\delta = 1000000000000066600000000000001$ (prime by the way). Whenever $|f(y) - f(x)| = |c - c| = 0 < \varepsilon$

(b) Fix arbitrary y, let $\varepsilon > 0$ and let $\delta = \varepsilon$. Whenever $|y - x| < \delta$,

$$|f(y) - f(x)| = |y - x| < \delta = \varepsilon$$

(c) We do induction on the degree of n. The base case is when $n = 0$ which is just the constant polynomial. From part (a), we know that this is continuous. For the inductive step, we assume that a k degree polynomial is continuous. Let this polynomial be $P_k(x)$. We must now prove that a $k+1$ polynomial is continuous. Let this polynomial be $P_{k+1}(x)$. We know that

$$P_{k+1}(x) = a_{k+1}x^{k+1} + P_k(x).$$

Now for any x_0, using limit properties, we have

$$\begin{aligned} \lim_{x\to x_0} P_{k+1}(x) &= \lim_{x\to x_0} a_{k+1}x^{k+1} + P_k(x) \\ &= a_{k+1} \lim_{x\to x_0} x^{k+1} + \lim_{x\to x_0} P_k(x) \\ &= a_{k+1}x_0^{k+1} + P_k(x_0) = P_{k+1}(x_0) \end{aligned}$$

Therefore $P_{k+1}(x)$ is continuous. (The second to last step follows from the fact that $P_k(x_0)$ and x^{k+1} is continuous).

(4) For the sake of contradiction, let us assume that this function is continuous and $x = 0$. For all $\varepsilon > 0$, there exists a $\delta > 0$ such that $|x| < \delta$,

$$|f(x) - f(0)| = \left|\frac{1}{x} - r\right| < \varepsilon.$$

Since $1/x$ increase as $|x|$ gets smaller, for any rational r, we can always pick ε such that we will never be able to pick a δ.

(5) For all $\varepsilon_1 > 0$, there is a $\delta_1 > 0$ such that $|x - x_0| < \delta_1 \implies |f(x) - L| < \varepsilon_1$ for all real x. Similarly, for $\varepsilon_2 > 0$, there is a δ_2 such that $|x - x_0| < \delta_2 \implies |g(x) - D| < \varepsilon_2$ for all real x.

1. $cf(x) \to cL$ means that for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all x, whenever $|x - x_0| < \delta$

$$|cf(x) - cL| = |c||f(x) - L| < \varepsilon \implies |f(x) - L| < \varepsilon/|c|.$$

   Now we pick δ such that it is equal to δ_1 when $\varepsilon_1 = \varepsilon/|c|$.

2. For this case, we have

$$|f(x) + g(x) - L - D| < |f(x) - L| + |g(x) - D|.$$

   Now we choose $\delta = \max(\delta_1, \delta_2)$ so whenever $|x - x_0| < \delta$,

$$|f(x) - L| + |g(x) - D| < \varepsilon_1 + \varepsilon_2.$$

   Notice that ε_1 and ε_2 can be any positive real so we can always make $\varepsilon_1 + \varepsilon_2 \le \varepsilon$ so we are done.

(10) Let $\lim_{x\to 0} \sin(1/x) = L$. Therefore for all $\varepsilon > 0$, there exists $\delta > 0$ such that whenever $|x - 0| = |x| < \delta$,

$$|\sin(1/x) - L| < |\sin(1/x)| + |L| < 1 + |L| < \varepsilon$$

which is not true for any $\varepsilon \ge 1$.

(11)

(a) This inequality is clearly true when $n = 0$. Suppose it is true when $n = k$. We have $(1 + x)^k \ge 1 + kx$. Multiplying both sides my $1 + x$ gives us

$$(1 + x)^{k+1} \ge (1 + kx)(1 + x) = (1 + kx) + (1 + kx)x \ge 1 + (k + 1)x.$$

Therefore by induction, we are done.

(b) We know that

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!} \implies e = \sum_{j=0}^{\infty} \frac{1}{j!}$$

so (y_n) is bounded above.

(c) We know that

$$x_n = \left(1 + \frac{1}{n}\right)^n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \le \sum_{k=0}^{n} \frac{1}{k!} = y_n.$$

Since $x_n < y_n$ and (y_n) is bounded, we are done.

5 Week 5

(1) Let $x \in (a, b)$. Choose $\varepsilon < \min(d(x, a), d(x, b))$. We know $\varepsilon > 0$ since a and b is not in (a, b) so $d(x, a), d(x, b) > 0$. We can see that $d(x, y) < \varepsilon$ implies that $y \in (a, b)$ so (a, b) must be open

For $[a, b]$, we choose the same ε and see that $\mathbb{R}\backslash[a, b]$ is open. Therefore $[a, b]$ must be closed.

(2) The set

$$\cap_{k=1}^{n}(-1/k, 1/k)$$

is the $\{0\}$ which is closed.

(3) Let S be a finite set and $x \in X\backslash S$. Choose

$$\varepsilon < \min(d(x, y_1), ..., d(x, y_n))$$

where each $y_i \in S$. We know that ε can be positive since each y_i is not in $X\backslash S$. Choosing this ε makes it such that $d(x, z) < \varepsilon \implies z \in X\backslash S$ so S must be closed.

Assume some $x \in S$ is a limit point. This would mean that every open neighborhood of x contains elements of S. Since S is a finite set, we can always make the open ball centred at x small enough such that it does not contain an element of S which is a contradiction.

(4) Consider the following sequence: We start with a point inside an open ball around x. For each term in the sequence we take a point in an open ball smaller than the open ball for the previous term. We can always do this since x is a limit point. Therefore the terms get closer and closer to x which means that they converge to x. Specifically, there is an integer N such that $n > N \implies |x_n - x| < \varepsilon$ for any ε.

(5) Assume S is not $\varnothing$ or $\mathbb{R}$ but clopen. Since it is open, it must be in the form (a, b) but this is not closed since $\mathbb{R}\backslash(a, b)$ is not open. Therefore we have a contradiction.

(6) Let $S = \{1/n : n \in N\} \cup 0$ and consider an open cover C of S. Now let S' be the set of all elements of S that are in at least 2 elements of C. We know that S' is a finite set since S is a discrete countably infinite set. Therefore we can replace these intersections with one set and we will have a finite subcover.

6 Week 6

(1)

(a) The function $f : \mathbb{R} \to \mathbb{R}$ satisfying

$$\begin{cases} -1 & x < 0 \\ 1 & x \geq 0 \end{cases}$$

works since for any $S \subseteq \mathbb{R}$, we know that $f(S)$ is either $\{1, -1\}, \{1\}$, or $\{-1\}$ which are compact sets.

(2) Let $\varepsilon > 0$ be arbitrary and choose $\delta = \varepsilon$. When $x, y \in [-1, 1]$, we have $x^2 < |x|$ and $y^2 < |y|$ so when $|x - y| < \delta$,

$$|x^2 - y^2| < |x - y| < \delta = \varepsilon$$

so we are done. However when $f : \mathbb{R} \to \mathbb{R}$, we cannot make this argument and δ has to depend of x.

(3)

(a) Let $\varepsilon > 0$ be arbitrary and choose $\delta = \varepsilon$. When $x, y \in [a, \infty)$, we have whenever $|x - y| < \delta$,

$$\left|\frac{1}{x} - \frac{1}{y}\right| = \frac{|y - x|}{|xy|} < |y - x| = |x - y| < \delta = \varepsilon$$

and since δ doesn't depend on x, we are done.

(b) We can see that

$$\bigcup_{a>0} [a, \infty) = (0, \infty).$$

Since f is uniformly continuous for all $x \in [a, \infty)$, we see that f must be uniformly continuous on $(0, \infty)$.

(5) Since f is Lipshitz we know there exists a $K > 0$ such that

$$\rho(f(x), f(y)) \leq K d(x, y)$$

for all $x, y \in X$. Let $\varepsilon > 0$ be arbitrary and $\delta < \varepsilon/K$. Now whenever $d(x, y) < \delta$,

$$\rho(f(x), f(y)) \leq K d(x, y) < \varepsilon$$

so f is uniformly continuous.

(6) The image of f must have at least $f(a)$ and $f(b)$. Assume these are the only two elements so $f(a) \neq f(b)$ since f is non-constant. Since f is continuous, for all $\varepsilon > 0$ and $x, y \in [a, b]$, there exists a positive δ such that whenever $|x - y| < \delta$,

$$|f(x) - f(y)| < \varepsilon.$$

However, $f(x)$ and $f(y)$ are either $f(a)$ or $f(b)$. Say we pick x and y such that $f(x) = f(a)$ and $f(y) = f(b)$. Then $|f(x) - f(y)|$ will be a nonzero constant since $f(a) \neq f(b)$. Therefore we will always be able to pick a ε such that

$$\varepsilon > |f(x) - f(y)|$$

so we have a contradiction.

(8)

(a) We know that $\frac{\sin x}{x}$ is continuous for all $x \neq 0$. Since $\lim_{x \to 0} \frac{\sin x}{x} = 1$ so it doesn't follow the intermediate value property.

(b) A counter example is $f(x) = 1/x^2$. All restrictions of $f|_[a, b]$ follow the intermediate value property but $f(x)$ is discontinuous at $x = 0$.

7 Week 7

(1) Since f' is bounded, we have $|f'(x)| < N$ for any x and some N. This implies that

$$\left|\frac{f(x) - f(x_0)}{x - x_0}\right| < N$$

since

$$f'(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

Let $\varepsilon > 0$ and choose $\delta = \varepsilon/N$. Whenever $|x - x_0| < \delta$,

$$|f(x) - f(x_0)| = |x - x_0| \cdot \left| \frac{f(x) - f(x_0)}{x - x_0} \right| < \delta \cdot N = \varepsilon.$$

(2) Let $f, g : (a, b) \to \mathbb{R}$ be differentiable functions, and $k \in \mathbb{R}$ a constant.

1.

$$\begin{aligned} \frac{d}{dx}(kf) &= \lim_{h\to 0} \frac{kf(x+h) - kf(x)}{h} \\ &= k \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= k\frac{df}{dx} \end{aligned}$$

2.

$$\begin{aligned} \frac{d}{dx}(f+g) &= \lim_{h\to 0} \frac{(f+g)(x+h) - (f+g)(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h) + g(x+h) - f(x) - g(x)}{h} \\ &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} + \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\ &= \frac{df}{dx} + \frac{dg}{dx} \end{aligned}$$

(3) Let $f(x) = x^2 \sin(x)$. By the product rule, we have

$$\frac{df}{dx} = \frac{d}{dx}(x^2)\sin(x) + x^2 \frac{d}{dx}(\sin(x)) = 2x\sin(x) + x^2\cos(x).$$

(4) We have

$$\begin{aligned} \frac{d}{dx}(\sin(x^2)) &= \lim_{h\to 0} \frac{\sin(x^2 + 2xh + h^2) - \sin(x^2)}{h} \\ &= \lim_{h\to 0} \frac{\sin(x^2)\cos(2xh + h^2) + \cos(x^2)\sin(2xh + h^2) - \sin(x^2)}{h} \\ &= \sin(x^2) \lim_{h\to 0} \frac{\cos(2xh + h^2) - 1}{h} + \cos(x^2) \lim_{h\to 0} \frac{\sin(2xh + h^2)}{h} \\ &= \sin(x^2) \lim_{h\to 0} \frac{\cos(2xh + h^2) - 1}{2xh + h^2}(2x + h) + \cos(x^2) \lim_{h\to 0} \frac{\sin(2xh + h^2)}{h}(2x + h) \\ &= 2x\cos(x^2) \end{aligned}$$

(5) We have

$$\frac{df}{dx} = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

8 Week 8

(1) Let
$$h(x) = f(x) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a)).$$
We see that $h(a) = h(b) = f(a)$ so we can use Rolle's Theorem: there must exist a c such that
$$0 = h'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)} g'(c).$$
Rearranging this gives us
$$(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$$
giving us our desired result.

(3) Let the extension be $g : [a, b] \to \mathbb{R}$. Consider a Cauchy sequence (x_n) that converges a. Because (x_n) is Cauchy, we know that $(f(x_n))$ converges to some real number. We define $g(a)$ as this real number. We can do the same thing for b to get $g(b)$. (I think there was an exercises we prove that we can find the limit at a point if we look at a sequence that converges to that point rather than looking at every x. We can use this to prove that g is continuous).

(5) We have that P_5 for $\sin(x)$ is
$$\sin(c) + \cos(c)(x - c) - \frac{\sin(c)}{2!}(x - c)^2 - \frac{\cos(c)}{3!}(x - c)^3 + \frac{\sin(c)}{4!}(x - c)^4 + \frac{\cos(c)}{5!}(x - c)^5$$

(6) We have that P_5 for $\cos(x)$ is
$$-\cos(c) - \sin(c)(x - c) - \frac{\cos(c)}{2!}(x - c)^2 + \frac{\sin(c)}{3!}(x - c)^3 + \frac{\cos(c)}{4!}(x - c)^4 - \frac{\sin(c)}{5!}(x - c)^5$$

(7) By definition, we have
$$P_k(x) = \sum_{j=1}^{k} \frac{P^{(j)}(c)}{j!}(x - c)^j$$
and
$$R_k(x) = P(x) - \sum_{j=1}^{k} \frac{P^{(j)}(c)}{j!}(x - c)^j.$$
Now we must calculate the following limit from $x \to c$ of
$$\begin{aligned}\frac{P(x)}{(x - c)^k} - \sum_{j=1}^{k} \frac{P^{(j)}(c)}{j!}(x - c)^{j-k} &= \frac{1}{(x - c)^k} \sum_{i=0}^{n} a_i x^i + \sum_{j=0}^{k} (x - c)^{j-k} \sum_{i=0}^{n} a_i \frac{i!}{j!(i - j)!} x^{i-j} \\ &= \frac{1}{(x - c)^k} \sum_{i=0}^{n} a_i x^i + \sum_{j=0}^{k} (x - c)^{j-k} \sum_{i=0}^{n} a_i \binom{i}{j} x^{i-j}\end{aligned}$$
which we can see is 0.

9 Week 9

(1) If $|x| < 1$, we have

$$\sum_{k=1}^{\infty} x^k = \frac{x}{1-x}$$

so the series converges. Now let us prove the converse. Suppose the series converges onto N but $|x| \geq 1$. The convergence says

$$\lim_{n\to\infty} \sum_{k=1}^{n} x^k = \lim_{n\to\infty} \frac{x - x^{n+1}}{1-x} = N.$$

However, this is a contradiction when $|x| \geq 1$ since $x^{n+1} \to \infty$ as $n \to \infty$ so the limit doesn't exist. Therefore when the series converges, we know that $|x| < 1$.

(2) Let

$$a_k = \frac{1}{k \log_2^{\alpha}(k)}.$$

Therefore

$$\sum_{k=1}^{\infty} 2^k a_{2^k} = \sum_{k=1}^{\infty} \frac{1}{k^{\alpha}}$$

which converges by the p-series test since $\alpha > 1 \implies \alpha+1 > 1$. Therefore by Cauchy's Condensation Test, we have that our original series converges.

(3) We know that

$$n^{1/\ln(n)} = e^{\ln(n)^{1/\ln(n)}} = e$$

so the limit is just e.