Problem Set Solutions for

Point-Set Topology

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Contents

1 Week 1

(1) We first start with $\mathcal{T} = \{\varnothing, X\}$. It follows the first property since $\varnothing$ and X are both in $\mathcal{T}$. The second property says that $\varnothing \cup X = X$ must be in $\mathcal{T}$ which is true. Finally the third property says that $\varnothing \cap X = \varnothing$ is in $\mathcal{T}$ which is also true.

Now let $\mathcal{T} = \mathcal{P}(X)$. The first property follows by definition. For the second property, if we take the finite union of elements of $\mathcal{T}$, we get subsets of X which by definition are in $\mathcal{P}(X) = \mathcal{T}$. In the same manner, if we take the intersection of two elements of $\mathcal{T}$, they must be in $\mathcal{T}$ since they are subsets of X. Therefore $\{\varnothing, X\}$ and $\mathcal{P}(X)$ are both topologies on X.

(2) Let $X = \{a, b, c\}$. Let $\mathcal{T} = \{\varnothing, X\} \cup Y$ and we must find the number of sets $Y \subseteq \mathcal{P}(X) \backslash \{\varnothing, X\}$ that make $\mathcal{T}$ a topology on X. We do casework on the number of elements in Y.

- If Y has 0 elements, there is only one possibility, $Y = \varnothing$, which works by the first problem giving us a total of **1** for this case.
- If there is 1 element in Y, it can be any subset of X that isn't $\varnothing$ or X giving us a total of **6** for this case.
- If there are 2 elements in Y, one of the elements needs to be a singleton and the other needs to have a size of 2. If we have any other combination, property (2) or (3) won't be satisfied. This gives us a total of $3 \cdot 3 = \mathbf{9}$ for this case.
- If there are 3 elements in Y, we need to have two of the elements of size 2 and the other one being the intersection of these two elements, which is a singleton. There are

  $$\binom{3}{2} \cdot 1 = 3$$

  ways of ding this.

  However, we could also have one size 2 subset and two singletons where their union is the size 2 subset. There are 3 ways to pick the size 2 subset so this gives us a total of $3 + 3 = \mathbf{6}$ for this case.
- If there are 4 elements in Y, consider the first subcase in the previous case where we picked two size 2 subsets and one singleton. We can pick any of the other singletons as our fourth element to give us a total of $3 \cdot 2 = \mathbf{6}$ for this case.
- It is impossible for Y to have 5 elements so we have a total of **0** for this case.
- Finally, there is only **1** way for Y to have 6 elements.

Adding all our cases together gives us $1 + 6 + 9 + 6 + 6 + 0 + 1 = \boxed{29}$ topologies.

(3) First we must prove that $\mathcal{B} = a\mathbb{Z} + b$ is a topological basis by proving that it follows the two properties:

1. Let x be an integer. We know that $x \in a\mathbb{Z} + x \in \mathcal{B}$.
2. Let $B_1 = a_1\mathbb{Z} + b_1$ and $B_2 = a_2\mathbb{Z} + b_2$. We see that

   $$B_1 \cap B_2 = \operatorname{lcm}(a_1, a_2)\mathbb{Z} + b_1 \in \mathcal{B}.$$

Next we prove that $\mathcal{B}$ generates the evenly spaced integer topology. However, this is trivial since we *defined* that the evenly spaced integer topology is the set of unions of the elements in $a\mathbb{Z} + b = \mathcal{B}$.

(4) Let us call $\mathcal{B}$ be the open balls of (X, d). We can prove that $\mathcal{B}$ is a topological basis by proving that it follows both properties:

1. For all $x \in X$, we can create a open ball $B \in \mathcal{B}$ centered at x. Therefore, every x is in $B \in \mathcal{B}$.
2. Let $x \in B_1 \cap B_2$ for some $B_1, B_2 \in \mathcal{B}$. Now let $B_3 \in \mathcal{B}$ be an open ball centered at x with radius r. By definition, this means that $x \in B_3$. Because open balls are open, there exists a r such that $B_3 \subseteq B_1 \cap B_2$ meaning this property is satisfied.

Now we must prove that $\mathcal{B}$ generates the topology induced by d which we can call $\mathcal{T}$. Recall that $\mathcal{T}$ contains the subsets where there always exists an open ball in the set. First notice that open balls of any radius are open as per this definition so $\mathcal{B} \subseteq \mathcal{T}$. Therefore by property (2) of topologies, we know that any union of elements of $\mathcal{B}$ must be in $\mathcal{T}$ so we are done.

(5) We can see that $\mathcal{T}$ follows the first property since $\mathcal{T}$ is a collection of subsets so it must contain all of X. For the second property, by property 3 in the definition of topologies, we know that $B_1 \cap B_2$ is in $\mathcal{T}$. Therefore $\mathcal{T}$ is a topological basis and it generates itself by the second property of the definition of topologies. This means that we are done.

(6) If $\mathcal{B}$ is the topological basis for $\mathcal{T}$, then $\mathcal{T}$ is generated by $\mathcal{B}$ so $U = B_1 \cup B_2 \cup \cdots \cup B_k$ for some $B_1, B_2, ..., B_n \in \mathcal{B}$. Since $x \in U$, there must be some B_k that contains x. This B_n is also a subset of U by definition so the claim is proven.

For the opposite direction, we must prove that finite unions of the elements of $\mathcal{B}$ create $\mathcal{T}$. In other words, we must prove that $U = B_1 \cup B_2 \cup \cdots \cup B_k$ for some $B_1, B_2, ..., B_n \in \mathcal{B}$. We are given that for every element in U, there exists a $B \in \mathcal{B}$ that contains the element and is also a subset of U. Let S be the set of all of these B's for every element of U. Now we see that the union of all the elements of S must be U which finishes our proof.

(7)

(a) For the first property, the set of all open intervals must contain all of $\mathbb{R}$. For the second property $(a, b) \cap (c, d)$ is either (a, b), (c, d), (c, b), or (a, d) which are all in $\mathcal{B}$.

(b) The topology induced by the Euclidean Metric is the set of all subsets $U \subseteq \mathbb{R}$ such that whenever $x \in U$, then $B_\varepsilon(x) \subseteq U$ for some $\varepsilon > 0$. However, we know that open balls in $\mathbb{R}$ are just open intervals. Therefore, this induced topology is precisely the same as $\mathcal{B}$.

(10) Let $\mathcal{T}$ be generated by $\mathcal{B}$. For the sake of contradiction let $\mathcal{B} \subseteq \mathcal{T}'$ be a topology smaller than $\mathcal{T}$. By the second property of topologies, we know that $\mathcal{T}'$ must contain all finite unions of elements of $\mathcal{B}$ since $\mathcal{B} \subseteq \mathcal{T}'$. By the definition, this means that $\mathcal{T}' = \mathcal{T}$ which contradicts the statement that $\mathcal{T}'$ is smaller than $\mathcal{T}$.

2 Week 2

(1) Let $f : \mathbb{R} \to \{1\}$ and let $A = \{x\}$ and $B = \{y\}$ and where $x \neq y$ and $x, y \in \mathbb{R}$. Therefore, we have $A \cap B \neq \varnothing$. However $f(A) \cap f(B) = \{1\} \cap \{1\} = \{1\}$. This means that $f(A \cap B) \not\subseteq f(A) \cap f(B)$

(2) This works like normal functions where $f(f^{-1}(V)) = V$ and $f^{-1}(f(U)) = U$. However, the reason there are subsets is when $f(V) = \varnothing$ or $f^{-1}(U) = \varnothing$.

(3) If f is continuous, then $f^{-1}(U)$ is open in A for all open sets $U \subseteq B$. Now every closed set in B can be expressed as $V = B \setminus U$ where U is again all open sets in B. Now

$$f^{-1}(V) = f^{-1}(B \setminus U) = A \setminus f^{-1}(U)$$

which is closed since $f^{-1}(U)$ is open.

On the other hand, assume $f^{-1}(V)$ closed whenever V is closed. Then every open set in B can be expressed as $U = B \setminus V$. This means that

$$f^{-1}(U) = f^{-1}(B \setminus V) = A \setminus f^{-1}(V)$$

which is open since $f^{-1}(V)$ is closed. Therefore f is continuous

(4) If f is continuous, then $f^{-1}(V)$ for open V is open in X. Now for each open neighborhood V of $f(x)$, we know that $f^{-1}(V)$ is a open neighborhood of x since it contains x and it is open. Now by Proposition 2.2, we have $f(f^{-1}(V)) \subseteq V$ so we are done.

(5) Let $V \in Z$ be open. Then $g^{-1}(U)$ must be open in Y. This means that $f^{-1}\left(g^{-1}(V)\right)$ must be open in X which means that we are done.

(6) IF f is continuous, then $f^{-1}(V)$ is open in X for all open sets $V \subseteq Y$. Now we can express V as a union of elements of $\mathcal{B}$ so if A is some set and $\{B_\alpha\}_{\alpha \in A} \subseteq \mathcal{B}$

$$f^{-1}(V) = f^{-1}\left(\bigcup_{\alpha \in A} B_\alpha\right) = \bigcup_{\alpha \in A} f^{-1}(B_\alpha)$$

so all $f^{-1}(B_\alpha)$ are open. Going through open V tells us that all $B \in \mathcal{B}$ are open.

(7) We must prove that the cofinite topology follows all three properties of topologies. Let $x \in \mathbb{R}$. Now we must prove that there exists a $U \in \mathcal{T}$ such that $x \in U$. We can choose $U = \mathbb{R} \setminus \{y\}$ where $y \neq x$. Since a singleton is finite, we have $U \in \mathcal{T}$. We can also see that $x \in U$ so the topology follows the first property.

Let A and B be any two elements of our topology. Let $A = \mathbb{R} \setminus A'$ and $B = \mathbb{R} \setminus B'$ where A' and B' are both finite sets. Now we have $A \cup B = \mathbb{R} \setminus (A' \cap B')$ and since $A' \cap B'$ is finite, we have $A \cup B \in \mathcal{T}$. We can also see that $A \cap B = \mathbb{R} \setminus (A' \cup B')$ and since $A' \cup B'$ is finite, we have $A \cap B \in \mathcal{T}$.

(8) We have proven that the standard topology is the same as the one induced by the Euclidean metric. However, this topology is just the set of all open intervals which can be formed from countable unions of disjoint intervals. Therefore we are done.

(9) Let $\mathcal{T}_{YZ}$ be the topology induced by Y and $\mathcal{T}_{XZ}$ be the topology induced by X. We can see that

$$\begin{aligned}
\mathcal{T}_{YZ} &= \{U \cap Z : U \in \mathcal{T}_Y\} \\
&= \{(U \cap Y) \cap Z : U \in \mathcal{T}\} \\
&= \{U \cap (Y \cap Z) : U \in \mathcal{T}\} \\
&= \{U \in Z : U \in \mathcal{T}\} \\
&= T_{XZ}
\end{aligned}$$

and we are done.

(10) Let $U_Y \in \mathcal{U}_Y$ be $U \cap Y$ for some $u \in \mathcal{T}$. Since U is open, it can be expressed as

$$U = \bigcup_{\alpha \in A} B_\alpha$$

where A is some set and all $B_\alpha \in \mathcal{B}$. Therefore

$$U_Y = U \cap Y = \left(\bigcup_{\alpha \in A} B_\alpha\right) \cap Y = \bigcup_{\alpha \in A} (B_\alpha \cap Y).$$

However, we know that $B_\alpha \cap Y \in \mathcal{B}_Y$ so $\mathcal{B}_Y$ generates $\mathcal{T}_Y$.

3 Week 3

(1) Let the topologies of X_1 and X_2 be $\mathcal{T}_1$ and $\mathcal{T}_2$, respectively. To prove that $X_1 \times X_2$ is homeomorphic to $X_2 \times X_1$, we must find a continuous bijection, $f : X_1 \times X_2 \to X_2 \times X_1$, such that f^{-1} is continuous.

Consider the map where we switch the values in each tuple. We know that f is surjective since the range is equal to the codomain. Additionally we know that f is injective since each element of the range corresponds to only one element in the domain so f is a bijection.

To prove that f is continuous, let $V \subseteq X_2 \times X_1$ be open. This means that $V = A \times B$ for some $A \in \mathcal{T}_2$ and $B \in \mathcal{T}_1$. Therefore $f^{-1}(V) = B \times A$ which must be open in $X_1 \times X_2$.

To prove that f^{-1} is continuous, let $U \subseteq X_1 \times X_2$ be open. This means that $U = A \times B$ for some $A \in \mathcal{T}_1$ and $B = \mathcal{T}_2$. Therefore $f(U) = B \times A$ which must be open in $X_2 \times X_1$.

(2) Let $Y = \{(x, x) \mid x \in X\}$. First we prove that the diagonal map $f : X \to Y$ is a bijection. First we see that the range covers the codomain so the function is surjective. Next we can see that there exists no two different elements of x that map to the same element which means that f is injective. Therefore f is bijective.

Next we prove f is a continuous function. Let the topology of X be $\mathcal{T}$ so the topology of Y is the subspace topology of the product topology. Let $V \subseteq Y$ be open which means that $V \in \{Y \cap (A \times B) \mid A, B \in \mathcal{T}\}$

(3) Since p is continuous, for every open subset $U \subseteq Y$, we have $f^{-1}(U)$ is open in X. If p is not a quotient map, then we must have some open set in X map to a closed set in Y. I suspect that p follows $p : \mathbb{R}^2. \to \mathbb{R}$ with the topologies of the domain and co domain being induced by the Euclidean metric because in some way, $\mathbb{R}^2$ has more open sets than $\mathbb{R}$.

(4) We claim that the map f from the quotient set to the circle as taking the unique number in our equivalence class that is in the range $[0, 2\pi)$ in our quotient space and letting that be the angle in our circle. We can easily see that this is a bijection.

Let $\mathcal{T}$ be the topology of $R/\sim$ and let $\mathcal{T}'$ be the subspace topology of the circle under the standard topology. Let $U \in \mathcal{T}'$. Now $f^{-1}(U)$ is the set of all equivalence classes of the angles of points in U. Therefore the this set of equivalence classes must be open by the definition of quotient topology. This proves that f is continuous. We can similarly prove that f^{-1} is continuous.

(5) This equivalence relation says that every element of $[0, 1]$ is equivalent to nothing except for itself and except 0 and 1 which are equivalent. This means that we can easily see that the relation is symmetric assuming $1 \sim 0$. By definition, we also see that $\sim$ is reflexive. Now if $a \sim b$ and $b \sim c$, then $b = a$ meaning that $a \sim c$. This completes our proof of $\sim$ being an equivalence relation.

Now we can think of the quotient space from the previous problem as the interval $[0, 2\pi)$ by looking at the unique number in each equivalence class that is in the range $(0, 2\pi]$. We can also think of $X/\sim$ as $[0, 1)$. Therefore we can stretch without tearing one of the intervals to form the other interval making the spaces homeomorphic.

(6) The relation is not an equivalence relation since it is not reflexive (by definition either the x or y coordinate needs to be different)

We can think of $[0,1]\times[0,1]/\sim$ as gluing the equivalent sides of the square together. This gives us a torus with is homeomorphic to all objects with one hole like a coffee mug.

(7) Let $\pi_1 : X\times Y \to X$ be a projection. We already know that π_1 is continuous. Let $U \subseteq X\times Y$ be open. This means that $U = A_1\times B_1 \cup A_2\times B_2 \cup \cdots$ by the definition of product topology. Now we have

$$f(U) = f(A_1\times B_1)\cup f(A_2\times B_2)\cup\cdots = A_1\cup A_2\cup\cdots.$$

We know that the A_i are open in X so $A_1\cup A_2\cup\cdots$ is also open in X. Therefore $f(U)$ is open in X so f^{-1} is continuous. This means that a projection is a homeomorphism if f is a bijection.

(8) The first two are not homeomorphic since the second has more connected parts than the first. the first and third are not homeomorphic since we cannot create a bijection. Finally, the second and third aren't homemorphisms since one has two connected parts while the other has one.

(9)

(a) Let $f(x) = y$ for any x. Therefore $y \in Y\times Z$ so $y = (a,b)$ for $a\in Y$ and $b \in Z$. Therefore we can define $f_1 : X\to Y$ and $f_2 : X \to Z$ such that the x's goes to a's in f_1 and the x's goes to b's in f_2

(b) If f is continuous, then for any open $V \subseteq Y\times Z$, then $f^{-1}(V)$ is open in X. Let $A\subseteq Y$ and $B\subseteq Z$ be open in their respective topological spaces. Therefore $A\times B$ is open in $Y\times Z$ so $f_1^{-1}(A)$ and $f_2^{-1}(B)$ are open in their respective topological spaces.

(10) First it is easy to see that reflections and translations are continuous and are bijections. Now since the inverse of a reflection and translation is also a reflection and translation, respectively, so the inverse of these transformations are also continuous. Therefore these transformations are homeomorphisms.

4 Week 4

(1) Let x be a real number. We can see that $x\in[x,b)$ for some b so the first property is satisfied. Next we can see the intersection of two elements of $\mathcal{B}$, our topological basis, is always in the form $[a,b)\in\mathcal{B}$. Therefore the second property also is satisfied.

(2) First we find an open cover of U. Since we want to find a tube around our line, we remove elements of the subcover that don't contain the line. Then we choose a finite number of these elements by projecting onto Y and evoking compacntess. Then we project onto X and take the intersection since we have a finite number of elements. This gives us our tube around the line and we are done.

(3) Let $\mathcal{O}$ be any open cover of A. This can be expressed as

$$\mathcal{O} = \{A\cap O : O \subset \mathcal{O}'\}$$

where $\mathcal{O}'$ is some open cover of X. Since every open cover of X can be converted into a finite subcover, $\mathcal{O}$ can be converted into a finite subcover meaning that A is compact.

(4) We first prove the forward direction. Assume that A does not contain the limit point x. Since A is closed, we know that $X\setminus A$ is open. Also x is in this open set so there exists an open neighborhood of x $U\subseteq X\setminus A$. However, this means that x is not a limit point since $U\cap A\neq\varnothing$. This raises a contradiction which means that A contains all its limit points

For the other direction, we get to assume that A contains all its limit points. Now assume that A is open. We can see that A doesn't contain the boundary limit points so A must be closed.

(5) I do not believe that such a proof exists.

(6) If X is compact, then every open cover has a finite subcover. Now let x be any element in X. We can notice that X is a neighborhood of x and it is compact so X is locally compact.

(7)

(a) We know that all closed and bounded subsets of $\mathbb{R}$ are compact. Consider the intervals of the form $[-x, x]$ which form $\mathbb{R}$.

(b) We can see that singletons are always compact since every open cover has the finite subcover of one of the open sets that contains the singleton. Therefore since the union of all singletons of the elements of the countable set form the countable set, this set is σ-compact

(c) Let $\mathcal{B}$ be the countable basis. However since X is locally compact, the basis elements are all compact so X must be σ-compact.

(8) We first impose a topology $\mathcal{T}$ on Y where $U \subseteq Y$ is open iff $U \subseteq X$ is open or if $U = Y \setminus K$ where K is compact in X. Let us prove that this is a topology. All elements in X are covered since $\mathcal{T}$ contains the compact subset so the complement of this must be open and it contains $U \setminus X$.

Next, unions of open sets of X, which are in $\mathcal{T}$, are open sets of X so also Y by definition. Additionally, the union of complements of compact subsets is complement of the intersection of the compact subsets which is therefore the complement of a compact subset and this is in $\mathcal{T}$.

Finally, the intersection of two open sets of X are in $\mathcal{T}$ by definition. Additionally, we can see that the intersection of two complements of compact subsets is a complement of some other compact subset (the union of the compact subsets). This finishes our proof that $\mathcal{T}$ is a topology.

Now we are ready to prove that Y is Hausdorff. Let x be some element of X. If $y \in Y$ is in X, we already know that we can pick two disjoint neighborhoods of x and y, so we let $y \in Y \setminus X$. Let U be an open neighborhood of x so U is contained in a compact subset that we call K. Now consider the set $Y \setminus K$. First we can see that $U \cap (Y \setminus K) = \varnothing$. Next, this set is open and contains y so it is an open neighborhood of y.

(9) The topology we are considering is all open sets of $\mathbb{R}$ and complements of compact subsets of $\mathbb{R}$. Since $\mathbb{R}$ is locally compact Hausdorff, we can just refer to the proof in problem (8). The only difference is the first property of topologies since there are two extra points this time; however, these are just contained in $[-\infty, \infty] \setminus [0, 1]$.

(10)

(a) Let $\mathcal{T}$ be the topology. Proving the first property of topologies is trivial. For open sets in $\mathcal{T}$ of the form (1), they follow the next two properties. Let U, V be open subsets of $\mathbb{R}$ containing 0, so $U \cup \{p\}$, $V \cup \{p\} \in \mathcal{T}$. We see that $(U \cup \{p\}) \cup (V \cup \{p\}) = (U \cup V) \cup \{p\} \in \mathcal{T}$ and $(U \cup \{p\}) \cap (V \cup \{p\}) = (U \cap V) \cup \{p\} \in \mathcal{T}$.

(b) The only open set containing $\{p\}$ is $U \cup \{p\}$ where U is an open subset of $\mathbb{R}$ containing 0. Therefore 0, p don't have any disjoint open neighborhoods.

(11) Since Y is the one point compactification of X, we know that Y is Hausdorff and compact and $Y \setminus X = \{y\}$ for some $y \in Y$. Now we know that all subspaces of Y must be Hausdorff so all we must show is that X is locally compact. Let $x \in X$ and let $U \subseteq X$ be an open neighborhood of X. Now we can append $\{y\}$ onto U and we get a compact set so we are done.

(12) Let $\mathcal{O}_X$ be an open cover of X. Since X is compact, there exists a finite subcover $\mathcal{O}'_X$ of $\mathcal{O}_X$. Now we define

$$\mathcal{O}_Y = \{f(O) : O \in \mathcal{O}_X\}$$

where is the homeomorphism. The elements of $\mathcal{O}_Y$ are open since homemorphisms map open sets to open sets. Therefore, we know $\mathcal{O}_Y$ is an open cover since

$$\bigcup_{U \in \mathcal{O}_Y} U = \bigcup_{V \in \mathcal{O}_\mathcal{X}} f(V) = f\left(\bigcup_{V \in \mathcal{O}_X} V\right) = f(X) = Y.$$

We can convert this open cover into a finite subcover by the same process as $\mathcal{O}_X \to \mathcal{O}'_X$. Thus Y is compact. We can prove the other way around similarly since f is a bijection.

5 Week 5

(1) Let C be a closed set containing A. When we calculate $\overline{A}$, we take the intersection of all closed sets containing A which includes C so $A \subseteq C$. Similarly, let U be an open set containing A. We know that A° is the union of all open sets containing A which includes U so $U \subseteq A^\circ$.

(2) We know that $\overline{A}$ is closed so $\overline{\overline{A}} = \overline{A}$. Similarly, we know that A° is open so $(A^\circ)^\circ = A^\circ$.

(3) Since A is closed in Y, we know that $Y \setminus A$ is open. Additionally, since Y is closed in X, we know that $X \setminus Y$ is open. Therefore, $X \setminus A = (X \setminus Y) \cup (Y \setminus A)$ which is open we are done.

(4)

- The relative interior of A in Y is the interior of $A \cap Y$ in Y. This is the largest open set in $A \cap Y$ which is just the interior of A in X or $A^\circ \cap X$.
- We defined ∂A as $\overline{A} \setminus A^\circ$ so this statement must be true by definition.
- Let $A = [0, 1]$ in $\mathbb{R}$. We know that $\overline{A} = [0, 1]$ and $A^\circ = (0, 1)$. Clearly $\overline{A} \not\subseteq A]^\circ$.

(5)

(a) Assume that X is connected. Additionally, assume that Y is not connected. This means that $Y = A \cup B$ where A, B are disjoint open subsets. Therefore $f^{-1}(A)$, $f^{-1}(B)$ are open in X and $f^{-1}(A) \cup f^{-1}(B) = X$. Now let $A' \subseteq f^{-1}(A)$ be an open subset of A and $B' \subseteq f^{-1}(B)$ be an open subset of B. This means that $A' \cup B' = C$ is open in X. Therefore $f(C) = f(A' \cup B') = f(A') \cup f(B')$ is open which is a contradiction since $f(A')$ and $f(B')$ is disjoint. Thus Y is connected. The converse can be proven in a similar manner since f is a bijection by the definition of homeomorphisms.

(b) Let $x, y \in X$. We know that there exist $p : [0, 1] \to X$ such that $p(0) = x$ and $p(1) = y$. Define $p' : [0, 1] \to Y$ (not the derivative) such that $p'(t) = f(p(t))$ where $f : X \to Y$ is a homeomorphism. Since f and p are continuous, p' must also be continuous so it is a path from $f(x)$ to $f(y)$. Since f is a bijection, every pair of points in Y must have a path between them

(c) Let S_1 and S_2 be two disjoint subsets of X such that $S_1 \cup S_2 = \overline{A}$. Since $A \subseteq \overline{A}$, there exists $S'_1 \subseteq S_1$ and $S'_2 \subseteq S_2$ where $S'_1 \cup S'_2 = A$. If S'_1 or S'_2 are not empty, then S'_1 and S'_2 are disjoint so we are done. If this is not true, either S'_1 or S'_2 is A but A can be open while S'_1 and S'_2 have to be closed so this raises a contradiction.

(6) Let U be an open set of $\mathbb{R}$ and let it be path connected. Therefore $f : [0, 1] \to \mathbb{R}^2$ and continuous exists. Here is the algorithm to convert a path into a horizontal-vertical path:

1. Go as vertical as you can towards the other point.
2. If you hit a boundary go horizontally towards the other point.
3. Keep switching until you hit the other point.

(7) (This problem was originally had $\mathbb{R}^4 \setminus (0,0,0,0)$). The only constraint in this problem is that our path cannot go through the origin. We can easily get around the origin by going to another point and going from there to our destination.

(8) We must prove that there exists no two disjoint open sets, A and B, such that $A \cup B = \mathbb{R}^\omega$ under the product topology. Let A and B be two disjoint open sets in $\mathbb{R}^\omega$. Let

$$A = \bigcup \prod_{i=1}^{\infty} U_i$$

and

$$B = \bigcup \prod_{i=1}^{\infty} V_i$$

where all but finitely many U_i and V_i are $\mathbb{R}$ and the rest are open intervals. If A and B are disjoint, then the individual basis elements that make them up are disjoint. If

$$\prod_{i=1}^{\infty} U_i \subseteq A, \quad \prod_{i=1}^{\infty} V_i \subseteq B$$

then there must exist at least one i such that $U_i \cap V_i = \varnothing$. Therefore $A \cup B$ cannot contain the whole $\mathbb{R}^\omega$

(9) Consider the map $f : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ where $f(x) = x/||x||$. Therefore if $x, y \in S^{n-1}$, since $\mathbb{R} \setminus \{0\}$ is path connected, there exists a path between $f^{-1}(x)$ and $f^{-1}(y)$. Let $p : [0,1] \to (\mathbb{R}^n \setminus 0)$ be this path. Now let $p' : [0,1] \to S^{n-1}$ (not the derivative) be a path such that $p'(t) = f(p(t))$. Since p and f are continuous maps, p' must also be a continuous map so this is our path from x to y.

6 Week 6

In this chapter, the definition of normal and regular got switched so "regular" means "normal" and "normal" means "regular" in the problems.

(1) If X is second countable, then it has a countable basis topology. We can use this as our countable collection of neighborhoods of some $x \in X$. By the definition of bases of topologies, we know that each neighborhood contains some $B \in \mathcal{B}$ so X must be first countable.

(2) We must prove that $\overline{\mathbb{Q}} = \mathbb{R}$. Let $\overline{\mathbb{Q}} = C$ so $\mathbb{Q} \subseteq C$ and C is closed. This means that $\mathbb{R} \setminus C$ is open so it is the union of intervals of the form (a, b). Therefore no rational number are in these intervals so $\mathbb{R} \setminus C = \varnothing$ so $C = \mathbb{R}$.

(3) We must first prove that intervals of the form $[x, x+1/n)$ follow the properties of topological bases. Let this bases be $\mathcal{B}$. We know that $\mathcal{B}$ covers $\mathbb{R}$ since for all $x \in \mathbb{R}$, we have $x \in [x, x+1/n)$. Next let $B_1 = [x_1, x_1+1/n)$ and $B_2 = [x_2, x_2+1/n)$. We know that $B_1 \cap B_2$ is either $\varnothing$, $[x_1, x_2+1/n)$, or $[x_2, x_1+1/n)$. All three of these are in $\mathcal{B}$ so this proves the second property. Finally, unions of these half intervals can produce any half interval so $\mathcal{B}$ is a basis of the half-open interval topology.

(4) Consider the topological basis

$$\mathcal{B} = \{(p, q) : p, q \in \mathbb{Q} \text{ and } p < q\}.$$

Let us first prove that $\mathcal{B}$ is a topological basis of the usual topology on $\mathbb{R}$. First for all $x \in \mathbb{R}$, there exists $p, q \in \mathbb{Q}$ such that $p < x < q$ so $x \in (p, q) \in \mathcal{B}$. Next, let $B_1 = (p_1, q_1)$ and $B_2 = (p_2, q_2)$. We know that $B_1 \cap B_2$ is $\varnothing$, (p_1, q_1), or (p_2, q_1) which are all in $\mathcal{B}$. Therefore, $\mathcal{B}$ follows both properties of topological bases. Finally, we can easily see that a general open set in the usual topology can be expressed as the union of elements of $\mathcal{B}$. Now $\mathcal{B}$ is a basis of the usual topology and also countable since the rationals are countable so $\mathbb{R}$ is second-countable. By Theorem 6.1, we know that $\mathbb{R}$ must be Lindelöf.

(5)

(a) If X is regular then it follows $\mathbf{T_0}$ and $\mathbf{T_3}$. Now for two elements $x_1, x_2 \in X$, by $\mathbf{T_0}$, there exists an open set U such that U contains either x_1 or x_2, say x. This means that $X \setminus U$ is closed and doesn't contain x. This means that by $\mathbf{T_3}$, there exist two disjoint open neighborhoods of x_1 and the closed set. This open neighborhood of the closed set is also an open neighborhood of x_2 since $x_2 \in X \setminus U$. Therefore X is Hausdorff.

(b) X is regular so it is $\mathbf{T_1}$ and $\mathbf{T_4}$. Let x be point and C be a closed subset. By $\mathbf{T_1}$, we know that x is a closed subset so we can apply $\mathbf{T_4}$. This gives us that there exist disjoint neighborhoods of x and C which means that X is $\mathbf{T_3}$. Let $a, b \in X$ be two points. By $\mathbf{T_1}$, they are closed. Now by $\mathbf{T_4}$, there exist disjoint neighborhoods of a and b so X is $\mathbf{T_0}$. Thus X must be normal.

(c) If X is Hausdorff, then there exist disjoint neighborhoods of any two points, say a and b. If U_a and U_b are the neighborhoods of a and b, respectively, then we can see that $a \in U_b$ and $b \in U_a$ since $U_a \cap U_b = \varnothing$. Therefore X is $\mathbf{T_1}$.

(d) Since X is $\mathbf{T_1}$, all singletons are closed so the closure of $\{a\}$ is not equal to the closure of $\{b\}$. Therefore X is $\mathbf{T_0}$

(6) If X is regular, then it is $\mathbf{T_4}$ and $\mathbf{T_1}$. Let A be a closed subset and let U be an open neighborhood of A. We know that $X \setminus U$ is closed so there exist disjoint neighborhoods of A and $X \setminus U$ by $\mathbf{T_4}$ which we call V_1 and V_2, respectively.

First we know that $X \setminus V_2 \subseteq U$ and $X \setminus V_2$ is closed. Now we let $V = (X \setminus V_2)^\circ$. We can confirm that $\overline{V} \subseteq U$ since $\overline{V} = X \setminus V_2 \subseteq U$. Next, since $V_1 \subseteq X \setminus V_2$ and V_1 is open, we know that $V_1 \subseteq (X \setminus V_2)^\circ = V$. Now V_1 contains A by definition so V must also contain A.

(7) Let (x_1, y_1) and (x_2, y_2) be two points in $X \times Y$. Let U_1 and U_2 be the two disjoint open neighborhoods of x_1 and x_2 and let V_1 and V_2 be the two disjoint open neighborhoods of y_1 and y_2. This means that $U_1 \times V_1$ and $U_2 \times V_2$ are two disjoint neighborhoods of (x_1, y_1) and (x_2, y_2), so $X \times Y$ is Hausdorff.

(8) We know that any two closed subsets contain the closure of the basis elements of the topology. However, these basis elements always have an element in common, namely the least common multiple of a, c. This means that there doesn't exist disjoint closed subsets so the space isn't normal.

(9)

(a) First we prove the forward direction. Let U be the open set that contains a but not b. (We define a and b to be two points in X). Now we know that $X \setminus U$ is closed so it is a superset of the closure of $\{b\}$. Since $X \setminus U$ doesn't contain a, the closure doesn't contain a so the closures of $\{a\}$ and $\{b\}$ must be different.

For the converse, let C and D be the closure of $\{a\}$ and $\{b\}$, respectively. First we see that a and b cannot be in $C \cap D$ or else $C \cap D$ would have been the closure of a and b. Now consider the open subset $U = (C \setminus (C \cap D))^\circ$. This set is a neighborhood of a but it is disjoint from D so it doesn't contain b. Now if $C = C \cap D$, then we can just set $U = C^\circ$ and X is still $\mathbf{T_0}$

(b) Since X is $\mathbf{T_1}$, it must also be $\mathbf{T_0}$, so the forward direction follows by definition. For the converse, assume that $\{a\}$ is not closed. This means that b could be in the closure of $\{a\}$ meaning $\overline{\{a\}} = \overline{\{b\}}$, This means that X is not $\mathbf{T_0}$ so there does not exist a U such that $a \in U$ and $b \notin U$ or $a \notin U$ and $b \in U$. This is a contradiction so we are done.

(10) Let $x \neq y$ be two points in X. By $\mathbf{T_{2.5}}$, there exist closed disjoint neighborhoods C and $D4$ of x and y, respectively. Now C° and D° are open neighborhoods of x and y and they must be disjoint since $C^\circ \subseteq C$ and $D^\circ \subseteq D$. Therefore X is Hausdorff.

7 Week 7

(1) Let A and B be disjoint closed subsets. Let $f : X \to [0,1]$ be a continuous function. We know that $A \subseteq f^{-1}\left(\left[0, \frac{1}{2}\right)\right)$ and $B \subseteq f^{-1}\left(\left(\frac{1}{2}, 1\right]\right)$ and the preimages are disjoint and open since f is continuous.

(2) First we can easily see that $\mathbb{R}^n$ is $\mathbf{T_1}$ under the usual topology. Additionally, for any two disjoint closed sets C and D in $\mathbb{R}^n$, we can easily see that there exists disjoint neighborhoods of C and D. Therefore $\mathbb{R}^n$ is $\mathbf{T_4}$. By Urysohn's lemma, since A and $\{x\}$ are closed, there must be a continuous map $f : \mathbb{R}^n \to [0,1]$ such that $f(A) = 0$ and $f(x) = 1$. This means that $\mathbb{R}^n$ is completely regular.

(3) Let A and $\{x\}$ be a closed set and singleton, respectively, in Y. Now we know that $A = Y \cap C$ for some closed C is X. We have $x \in Y$ and $x \notin B$ so $x \notin C$. This means that there exists a continuous map $f : X \to [0,1]$ such that $f(C) = 0$ and $f(x) = 1$. This means that the restriction $f|_Y$ is continuous and $f(A) = 0$ and $f(x) = 1$. Therefore Y is completely regular.

(4) If $x \in A$, then $d(x,x) = 0 \in \{d(x,y) : y \in A\}$ and since $d(x,y)$ is always positive, $D(x,A) = 0$. For the converse, we know that $d(x,y) = 0$ if $x = y$. Additionally, we know that $D(x,A) = 0$ if $d(x,y) = 0$ for some $y \in A$. This means that $x \in A$ and we are done.

(5) First we can see that $\rho(x,y) > 0$ since $|x_i - y_i| > 0$. Another easy property to check off is $\rho(x,y) = \rho(y,x)$ since $|x_i - y_i| = |y_i - x_i|$. Also if $\rho(x,y) = 0$, then $|x_i - y_i| = 0$ for all i so $x = y$. For the converse $x_i = y_i$ for all i so $\rho(x,y) = 0$.

Finally, we need to prove the triangle inequality. We know that $\rho(x,y) = \sup\{|x_i - y_i|\}$, $\rho(y,z) = \sup\{|y_i - z_i|\}$, and $\rho(x,z) = \sup\{|x_i - z_i|\}$. Next for any i, $\rho(x,y) \geq |x_i - y_i|$, $\rho(y, z_\geq |y_i - z_i|$, and $\rho(x,z) \geq |x_i - z_i|$. Next we know that $|x_i - y_i| + |y_i - z_i| \geq |x_i - z_i|$ so $\rho(x,y) + \rho(y,z) \geq \rho(x,z)$.

(6)

(a) For each U_i, notice that $X \setminus U_i$ is closed. Let x be any singleton in U_i. Since X is normal, there exist disjoint open neighborhoods around x and $X \setminus U_i$. Let the one around $X \setminus U_i$ be A so $X \setminus A$ is closed and $X \setminus A \subseteq U_i$. This means that if $U_i = (X \setminus A)^\circ$, then $\overline{V_i} \subseteq U_i$ and we are done.

(b) We can use the same process to get the W_i's. Now by Urysohn's lemma, there exists $\psi_i : X \to [0,1]$ such that $\psi_i(X \setminus V_i)$ is 0 and $\psi_i(\overline{W_i})$ is 1. This means that the support of ψ_i is inside $\overline{V_i}$

(c) By defining ϕ_i this way, the second property is satisfied. The first property is already satisfied from part (b) so the ϕ_i must be the partion of unity subordinate to $\{U_i\}$.

(7) We must prove that X is homeomorphic to a subspace of $\mathbb{R}^N$. First since X is compact, every open cover has a finite subcover. Since X is an n-manifold, every open set in X has an open set in $\mathbb{R}^n$ which is a homeomorphism. Therefore each element of the finite subcover $\{U_i\}$ is imbedded in $\mathbb{R}^n$ by $g_i : U_i \to \mathbb{R}^n$. Now let $\phi_1, \phi_2, ..., \phi_m$ be a partion of unity where ϕ_i is subordinate to U_i. Define $h_i = \phi_i \cdot g_i$ for $x \in U_i$ and $h_i = 0$ for all $x \notin \text{supp}(\phi_i)$

Then $F : X \to (\mathbb{R} \times \cdots \times \mathbb{R}) \times (\mathbb{R}^n \times \cdots \times \mathbb{R}^n)$ defined by $F(\phi_1, ..., \phi_m, h_1, ..., h_m)$ is an imbedding. This must be true since ϕ_i and h_i are continuous bijections so we are done

(8)

(a) We know that 0 and $e^{-1/x}$ are both continuous. Next, we see that $e^{-1/x}$ goes to 0 as x goes to 0 so f as a whole is continuous.

(b) When $x \leq 0$, then $g(x) = 0/(0 + e) = 0$. When $x \geq 1$, we have $g(x) = f(x)/f(x) = 1$.

(c) When $x \geq c$, then $c - x \leq 0$ so $g(c - x) = 0$ and when $x \leq c - 1$, then $c - x \geq 1$ so $g(c - x) = 1$.

(9) Let $\{U_i\}$ be the finite open cover of X and let $\{V_j\}$ be the open cover for Y. Now define

$$O = \{U_i \times V_j\}$$

to the the open cover for $X \times Y$. Next we claim that the map $F_{i,j} = \phi_i \cdot \psi_j$, where ϕ_i and ψ_j are partions of unities of X and Y, is a partion of unity for $X \times Y$.

First we see that $\text{supp}(\phi_i) \subseteq U_i$ and $\text{supp}(\psi_j) \subseteq V_j$. For the second property,

$$\begin{aligned} \sum F_{i,j} &= \sum_i \sum_j \phi_i \cdot \psi_j \\ &= \sum_i \phi_i \sum_j \psi_j \\ &= \sum_i \phi_i = 1 \end{aligned}$$

and we are done.

8 Week 8

(1) Let us do casework on the number of singletons in $\mathcal{F}$. If there are none, we can have one two-element subset in our filter since anymore will cause a singleton to be in our filter by property (2). By property (3), we see that $\{1, 2, 3\}$ must also be in $\mathcal{F}$. This is all we need for our filter so since there are three two-element subsets, this case adds 3 to our total.

If there is one singleton, we need to add all but one of the non-singletons except for $\varnothing$. We can do this for each singleton so this case adds 3 to our total.

Finally, if there are two singletons, this forces us to add all the nonempty subsets to our filter so this case addes 1 to our total. Therefore the total number of filters is $3 + 3 + 1 = \boxed{7}$

(2) First we can see that $\mathcal{F}$ does not contain ϕ since $\mathcal{B}$ does not contain ϕ.

Next let $F_1, F_2 \in \mathcal{F}$ and let $B_1, B_2 \in \mathcal{B}$ such that $B_1 \subseteq F_1$ and $B_2 \subseteq F_2$. We know that B_1 and B_2 cannot be disjoint since this would imply that $\varnothing \in \mathcal{B}$ which is not true. Therefore, $B_1 \cap B_2 \subseteq F_1 \cap F_2$ and since some $B_3 \in \mathcal{B}$ is in $B_1 \cap B_2$, we know that there exists a $B \in \mathcal{B}$ such that $B \subseteq F \subset G$ so $B \subset G$. Therefore $G \in \mathcal{F}$ and we are done.

(3) First we see that $\mathcal{F}$ does not contain ϕ since A is nonempty. Next we know that the intersection of any two elements must at least contain A. Therefore property (2) also holds. Finally, a superset of any element of the filter must also contain A so $\mathcal{F}$ must be a filter. Let $B \subset X$. If $B \supseteq A$, then $B \in \mathcal{F}$ but if not, we have $X \setminus \supseteq A$ so $X \setminus B \in F$. Therefore $\mathcal{F}$ is an ultrafilter.

(4) We know that $\bigcap_{F \in \mathcal{F}} \overline{F}$ is a closed subset of A since $\mathcal{F}$ contains A. Now if we just consider the closed subsets of A, we just get $\overline{A}$.

(5) Let $B_1, B_2, \in \mathcal{B}$ such that $f(A_1) = B_1$ and $f(A_2) = B_2$ for $A_1, A_2 \in \mathcal{F}$. Now let $B_3 = f(A_1 \cup A_2)$ so $B_3 = F(A_1 \cup A_2) \subseteq f(A_1) \cup f(A_2) = B_1 \cup B_2$. Therefore $\mathcal{B}$ is a fliterbase.

(6) Assume f is continuous at x and let $\mathcal{F}$ be a filter on X with $\mathcal{F} \to x$. Since f continuous at x, the preimage of any open neighborhood of $f(x)$ contains an open neighborhood of x. Let V be some open neighborhood of $f(x)$ and let $f^{-1}(V)$ contain U, an open neighborhood of x. We know that $f(U) \in \{f(F) : F \in \mathcal{F}\}$ which is a subset of $f(\mathcal{F})$ which is a subset of $f(\mathcal{F})$ so $f(U) \in f(\mathcal{F})$. This is all because $\mathcal{F} \to x$ so $U \in \mathcal{F}$. Additionally, we have $f(U) \subseteq V$ so V is generated by $f(U)$. Therefore, $V \in f(\mathcal{F})$ so $f(\mathcal{F}) \to f(x)$.

On the other hand, if $f(\mathcal{F}) \to f(x)$ whenever $\mathcal{F} \to x$, then $f(\mathcal{F})$ contains all open neighborhoods of $f(x)$. Let V be one of these. Since $V \in f(\mathcal{F})$, there exists an $A \in \mathcal{F}$ such that $f(A) \subset V$. This means that $A \subseteq f^{-1}(V)$. We claim that $x \in A$. If it were not, we could create an open neighborhood of x, which is also in $\mathcal{F}$, so that this neighborhood is disjoint from A. The only way to avoid this is for A to contain x. We just take the interior of A and we are done.

(7) Assume x is a limit point of $\mathcal{F}$. Consider the filterbase

$$\mathcal{B} = \{U \cap F : F \in \mathcal{F}, U \text{ is open, and } x \in U\}.$$

We first prove that $\mathcal{B}$ is a filterbase. Let $B_1 = U_1 \cap F_1$ and let $B_2 = U_2 \cap F_2$. Now

$$\begin{aligned} B_1 \cap B_2 &= (U_1 \cap F_1) \cap (U_2 \cap F_2) \\ &= (U_1 \cap F_1) \cap (F_2 \cap U_2) \\ &= U_1 \cap (F_1 \cap F_2) \cap U_2 \\ &= U_1 \cap F_3 \cap U_2 \\ &= U_1 \cap U_2 \cap F_3 \\ &= U_3 \cap F_3 \in \mathcal{B} \end{aligned}$$

where $F_3 = F_1 \cap F_2 \in \mathcal{F}$ and $U_3 = U_1 \cap U_2$. Now when we look at the filter generated by $\mathcal{B}$, we see that it is finer than $\mathcal{F}$ and it contains all open neighborhoods of x.

(8) If $\mathcal{G} \neq \mathcal{F}$ and $\mathcal{F} \subseteq \mathcal{G}$, then there exists a subset of X, A, that is in $\mathcal{G}$ but not in $\mathcal{F}$. Since $\mathcal{F}$ is an ultrafilter and $A \notin \mathcal{F}$, we know that $X \setminus A \in \mathcal{F} \subseteq \mathcal{G}$. However $\mathcal{G}$ is a filter so $X \setminus A$ and A cannot both be in $\mathcal{G}$. Therefore A cannot exist so $\mathcal{G} = \mathcal{F}$.

(10) Since $\mathcal{F}$ is an ultrafilter, either A or $X \setminus A$ is in $\mathcal{F}$. Now let $B \subset Y$. Now we can see that if B is not in $f(\mathcal{F})$, then there exists an element that generates $Y \setminus B$ so $f(\mathcal{F})$ is an ultrafilter.

9 Week 9

(1)

(a) We know that $\mathcal{P}$ is the set of all linearly independent subsets of $\mathbb{R}^3$. First $\subseteq$ is reflexive since $B \subseteq B$. If $A \subseteq B$ and $B \subseteq A$, then $A = B$ so $\subseteq$ is antisymmetric. Finally, $\subseteq$ is transitive since $A \subseteq B \subseteq C \implies A \subseteq C$.

(b) The maximal elements of $\mathcal{P}$ are the linearly independent sets that are not subsets of any other linearly independent sets. This is only the case with bases of $\mathbb{R}^3$.

(c) If an upperbound existed, then it would be the superset of the union of all linearly independent sets. However this will not be a linearly independent set since the upperbound would be the supoerset of a linearly dependent set. Therefore no upperbound exists.

(2)

- We can see that $N(z) \leq N(z)$ so $z \dot{\leq} z$. Next if $z \dot{\leq} w$ and $w \leq z$, then $N(z) \leq N(w)$ and $N(w) \leq N(z)$ so $N(z) = N(w)$. Finally, we can easily see that $\dot{\leq}$ is transitive since $\leq$ is transitive.
- This poset does not have an upperbound or maximal element since the norm can get arbitrarily big.
- First we can see that it isn't the case that $z \dot{<}$ since $z = z$. Since $\dot{\leq}$ is transitive, we can also see that $\dot{<}$ is transitive. Finally, we can trivially see that $\dot{<}$ is trichotomic.

(3) Let $\{X_\alpha\}_{\alpha \in A}$ be a countable collection of countable sets. Since this is a countable collection, we can write this collection as $\{X_n\}_{n \in \mathbb{N}}$. Let us consider random orderings of X_n for all $n \in \mathbb{N}$. Let x_{mn} be the mth element of X_n. We can write all the elements in matrix form where m corresponds to the row number and n corresponds to the column number:

$$\begin{matrix} x_{11} & x_{12} & \cdots \\ x_{21} & x_{22} & \\ \vdots & & \ddots \end{matrix}$$

Consider pick the elements in a diagonal fashion where we start from x_{11}, go down to x_{21}, go up right to x_{12}, go down to x_{22}, and so on. We can do this because of Axiom of Choice so the union must be countable.

(4) We can follow the same diagonal pattern as (3). Again we can do this only because of Axiom of Choice.

(5) Let $\mathcal{F}_0$ be a filter on X. Let S be the set of all filters containing $\mathcal{F}_0$ and consider the poset $(S, \subseteq)$. If C is a chain, we can see that $\bigcup F_i = \mathcal{F}$ where $F_i \in C$ is a filter. Therefore $\mathcal{F}$ is the upperbound of C so every chain has an upperbound. This means that by Zorn's lemma, there exists a maximal element G on S. Therefore, there exists an ultrafilter containing $\mathcal{F}$.

(6) We would not be able to use Tycnoff's Theorem since each X_α is not necessarily compact. This would mean that the product $\prod_{\alpha \in A} X_\alpha$ is not necessarily compact breaking the proof.

(7) Consider the binary operator $\dot{\leq}$ where $z \dot{\leq} w$ whenever $|z| < |w|$. We can essentially see that $\dot{\leq}$ must be a linear ordering since $<$ is a linear ordering. With this linear ordering equipped, we can create a topology in $\mathbb{C}$ just like $\mathbb{R}$: open balls. We define an open ball of radius $w \in \mathbb{C}$ centered around $z \in \mathbb{C}$ as

$$B_w(z) = \{x \in \mathbb{C} : x - z \dot{<} w\}.$$

Using these open balls, we can define the topology on $\mathbb{C}$ just like how we defined the topology on $\mathbb{R}$ and we are done.

(8) First we can see that $(a, b) \leq (a, b)$ since $a \leq a$ and $b \leq b$ so $\leq$ is reflexive. Next if $(a, b) \leq (c, d)$ and $(c, d) \leq (a, b)$, we know that $a \leq c$, $b \leq d$, $c \leq a$, and $d \leq b$. Therefore $a = c$ and $b = d$ so $(a, c) = (b, d)$. Finally, if $(a, b) \leq (c, d)$ and $(c, d) \leq (e, f)$, we have $a \leq c$, $b \leq d$, $c \leq e$, and $d \leq f$ so $a \leq e$ and $b \leq f$. Therefore $(a, b) \leq (e, f)$ which means that $\leq$ is a partial ordering. Additionally we can see that $<$ is a linear ordering since it is irreflexive, transitive, and trichotomic.

(9) We can easily see that $\leq$ is reflexive. However, $\leq$ is not symmetric since $(1, 2) \leq (2, 1)$ and $(2, 1) \leq (1, 2)$ but $(1, 2) \neq (2, 1)$. So, $\leq$ is not a partial ordering.

For $<$, we can easily see that $<$ is irreflexive. However $<$ is not transitive:

$$(1, 2) < (2, 1) \text{ and } (2, 1) < (1, 2)$$

but $(1, 2) \not< (1, 2)$ since $<$ is irreflexive. Therefore $<$ is not a linear ordering.

10 Week 10

(1)

(a) We can easily see that every point in $\mathbb{R}$ only has 1 output since it cannot be the case that $x = n$ and $|x - n| > 1$.

(b) By proposition 10.1, we must prove that for each x, the sequence $f_n(x)$ converges to 0 in $\mathbb{R}$. In other words, for every ε there is a N such that $n \geq N$ implies $|f_n(x)| < \varepsilon$. We see that $f_n(x) = 0$ in at least 2 steps so we can pick any $N \geq 2$.

(c) Assume it converged to g. Then for all open neighborhoods U of g, there is some N such that $n \geq N$ implies $f_n \in U$. We can see that there exists no g such that this happens if we consider the uniform topology. Therefore we have a contradiction.

(d) Since one topology converges f_n and the other doesn't, these two topologies must be defined.

(2) First we can easily see that this basis covers the whole space Y^X. Next let B_1 and B_2 be two basis elements:

$$B_1 = B_C(f, r_1) = \{g \in Y^X : \sup\{d(f(x), g(x))\} < r_1 \text{ for } x \in C\}.$$

Now we consider the intersection $B_1 \cap B_2$. Now we can see that the ball of radius

$$\frac{r_1 + r_2 - \rho(f, g)}{2}$$

is contained in $B_1 \cap B_2$.

(3) Since $f_n \to f$ on the CC-topology, for all ε, there exists an integer N such that $n \geq N$ implies $f_n \in B_C(f, \varepsilon)$. We define C as some compact subspace of X. Now the restriction won't do anything to the convergence in the CC topology. Therefore $f_n|_C$ converges uniformly to $f|_C$

(4) The RHS is telling us the elements in C that map to an element in V. We can find this by considering all elements that map to an element U and then seeing the ones that are in C. This is precisely what the LHS is doing so the LHS and RHS are equal.

(5) Let U be any open set on the uniform topology. We know that U is just some open ball using the uniform metric ρ. Let the center of U be f and the radius be ε. Now we can see that

$$U = \bigcup_C B_c(f, \varepsilon)$$

so U is in the CC topology. Therefore the uniform topology is coarser than the CC topology. If X is compact, the same holds since we can just make $C = X$ and the uniform topology will be a subset of the CC topology.

(6) We can use the same logic as the previous problem ans see that every element in the CC topology is in the topology of point-wise convergence. Therefore the CC topology is a subset of the topology of pointwise convergence so the CC topology must be coarser.

(7) Let us assume x is not in C. Since C is closed, we know that $X \setminus C$ is open. Because x is not in C, we have that $X \setminus C$ must be open so there must exist an N such that $n \geq N \implies x_n \in X \setminus C$. However, since $x_n \in C$ for all n, we have a contradiction so we are done.

(8) Let f be $f : \mathbb{R} \to \{0\}$ and let C be some compact subset in $\mathbb{R}$. By Hiene-Boral, we have $C = [a, b]$ for $a, b \in \mathbb{R}$. Without loss of generality, let $|a| > |b|$. Now let ε be any positive real and $N = |a|/\varepsilon$. Now we see that if $n \geq N$, then

$$\begin{aligned}\rho(f_n|_C, f_C) &= \sup\{|f_n(x)| : x \in C = [a, b]\} \\ &= \sup\left\{\left|\frac{x}{n}\right| : x \in [a, b]\right\} \\ &= \left|\frac{a}{n}\right|\end{aligned}$$

since $|a| > |b|$. Because $n > |a|/\varepsilon$, we know that $|n| > |a|/\varepsilon$ since $|a|/\varepsilon > 0$. Therefore $\varepsilon > |a/n|$ so

$$\rho(f_n|_C, f_c) < \varepsilon.$$

This proves that $f_n|_C \to f|_C$ so by proposition 10.3, we know that $f_n \to f$ on the CC topology. In other words, we have $f_n \to 0$.

Now if $f_n \to 0$ uniformly, then for all positive real ε, if $n \geq N$, then $\rho(f_n, f) = \sup\{|f_n(x)| : x \in \mathbb{R}\} < \varepsilon$. However the supremum is infinite so it cannot be less than some real number. Therefore $f_n \not\to 0$ uniformly.

(9) We follow the same logic as the previous problem. We use Proposition 10.3 again. Since C is compact, by Hiene-Boral, C must be in the form $[a, b]$. Because of this, the supremum is defined in the definition of the uniform metric for $f_n|_C$ and $f|_C$. However, if we consider the case of uniform convergence, the opposite happens: the supremum is infinite. Therefore, $f_n(x)$ converges in the CC topology but doesn't converge in the uniform topology.

(10) Since Y is Hausdorff, any two points can be separated by open sets. Now let f and g be two continuous functions from X to Y. Let y_1 and y_2 be two points in Y such that $d(y_1, y_2) = \rho(f, g)$. We can separate these two points in Y is Hausdorff. Now we can use the technique we use to separate f and g. Therefore, we are done.