Mathematics from Physics

Nandana Madhukara

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Agenda

The General Idea

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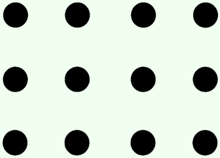
Physical Representation

- Ex: Commutative Property



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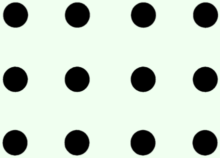
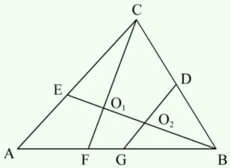


Using Physics

- Ex: Mass Points



Center of Mass Arguments

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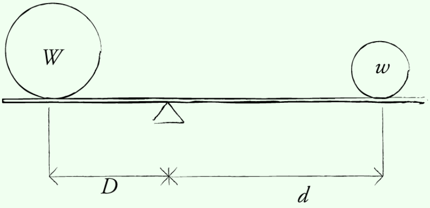
Axioms

1. Law of the lever

$$WD = wd$$



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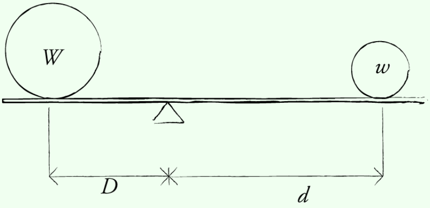

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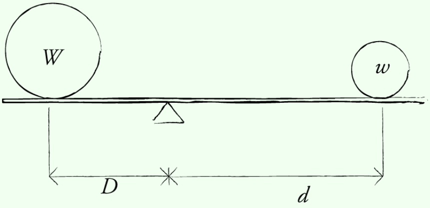

2. Every geometrical object has a CM
3. If C = CM of whole and c = CM of parts,

$$C = \sum c$$

Medians

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Theorem

The centroid of a triangle splits a median into two segments with the ratio of its lengths being 2:1

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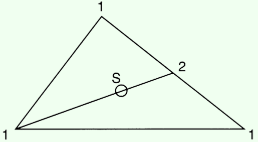
Proof.



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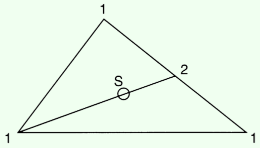

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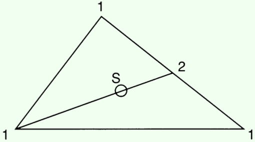

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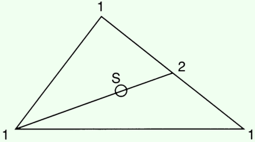

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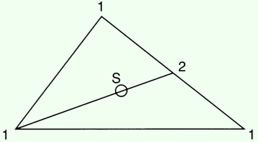

- Set all verticies to have mass of 1
- CM of one side (midpoint) will have mass of 2
- CM of triangle (centroid) is on median
- Centroid splits median into ration 2:1

□

Fermat Point

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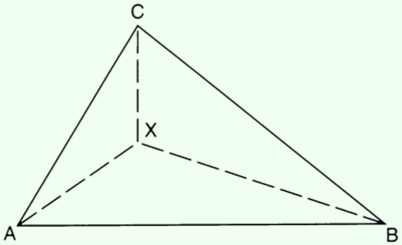
Question

For $\triangle ABC$, what is the point X such that $XA + XB + XC$ is minimized?



Fermat Point (cont.)

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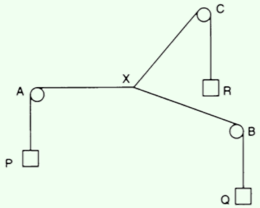
Proof.

- Consider the following pulley system with equal masses:



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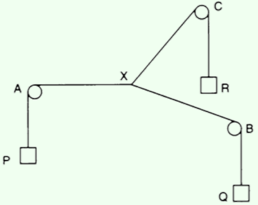

- X will move wherever the weights are balanced

Fermat Point (cont.)

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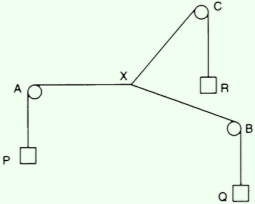

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- Happens when masses are as low as possible $\implies AP + BQ + CR$ is maximized

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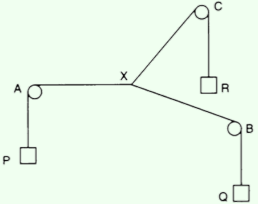

- X will move wherever the weights are balanced
- Happens when masses are as low as possible $\implies AP + BQ + CR$ is maximized
- Therefore $XA + XB + XC$ is minimized so X goes to the Fermat Point.

□

Pythagorean Theorem

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For a right triangle PQR, the sides satisfy

$$PQ^2 + QR^2 = PR^2.$$

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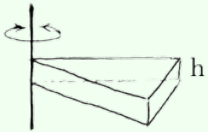
Proof.

- Consider the following setup with air inside



Pythagorean Theorem (cont.)

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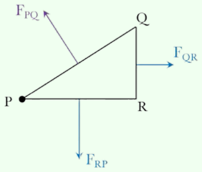
Proof (Cont.)

- The pressure will cause the following forces (aerial view)



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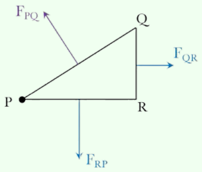

- Torques must cancel out:

$$F_{PQ} \times PQ/2 = F_{RP} \times PR/2 + F_{QR} \times QR/2$$

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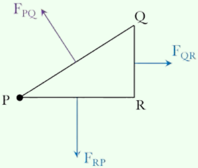

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- If pressure is p,

$$ph(PQ^2)/2 = ph(PR^2)/2 + ph(QR^2/2)$$

□

Multiplicative Scoring

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Game Rules

1. Start with a number n
2. Split n into i and $n - i$
3. Add $i(n - i)$ to the score
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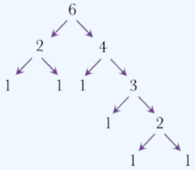

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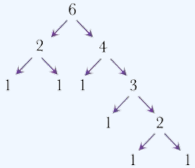

Score:

$$2 \times 4 + 1 \times 1 + 1 \times 3 + 1 \times 2 + 1 \times 1 = 15$$

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- However

$$\Delta PE = (n-1) + ... + 2 + 1 = \boxed{\frac{n(n-1)}{2}}$$

□

AM-GM

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Theorem

For weights $p_i \in [0, 1]$ *such that* $\sum p_i = 1$ *and numbers* a_i*,*

$$\sum p_i a_i \geq \prod a_i^{p_i}.$$

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- Let the weights be $p_i = m_i / M$ where $M = \sum m_i$
- Let initial temperature of ith mass be T_i
- If placed in thermal contact, final temperature will be

$$\overline{T} = \sum p_i T_i$$

AM-GM (cont.)

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Proof (Cont.)

- Over small interval of time, let temperature of ith mass change by dT_i' when it is at temperature T_i'

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- Entropy Change:

$$dS_i = \frac{dQ_i}{T_i'} = \frac{cp_i M dT_i'}{T_i'} = cp_i M d \ln T_i'$$

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- Total entropy change:

$$\Delta S_i = cM(p_i \ln \overline{T} - p_i \ln T_i) = cM(p_i \ln \overline{T} - \ln T_i^{p_i})$$

AM-GM (cont.)

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Proof (Cont.)

- Summing over all masses:

$$\Delta S = cM \left(\ln \overline{T} - \ln \prod T_i^{p_i} \right)$$

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Proof (Cont.)

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- Second Law of Thermodynamics: $\Delta S \geq 0$ so

$$\overline{T} = \sum p_i T_i \geq \prod T_i^{p_i}$$

□