Kinematics Team Problems Key

Fun With Fiziks

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Practice Problems Key

1. We first list the known and unknown information relevant to the problem to figure out what equation to use:

Knowns	Unknowns
$v_0 = 9\, m/s$	$a = ?\, m/s^2$
$\Delta x = 5\, m$	
$v_f = 0\, m/s$	

   We know that $v_f = 0\, m/s$ because Luke comes to a stop. We have v_0, Δx, v_f, and a, so we can use the equation $v_f^2 = v_0^2 + 2a\Delta x$ because it contains those variables. Solving for a,

$$a = \frac{v_f^2 - v_0^2}{2\Delta x} = \frac{0^2 - 9^2}{2 \cdot 5} = \boxed{-8.1\, m/s^2}$$

2. Here is the information we are given in the problem:

Knowns	Unknowns
$x_{J,0} = 0\, m$	$t = ?\, s$
$x_{L,0} = 10\, m$	
$v_{J,0} = 1\, m/s$	
$v_{L,0} = 2\, m/s$	
$a_J = 3\, m/s^2$	
$a_L = 2\, m/s^2$	

   We have information about v_0, a, t, and x_0, so we use $\Delta x = v_0 t + \frac{1}{2}at^2$. This might look different since we technically don't have Δx, but we can manipulate the equation to get

$$x(t) = x_0 + v_0 t + \frac{1}{2}at^2,$$

   since $\Delta x = x_f - x_0 = x(t) - x_0$. Let $x_J(t)$ and $x_L(t)$ represent Jonathan and Lilian's positions as a function of time, respectively. When Jonathan reaches Lilian, $x_J(t) = x_L(t)$. Then, $x_J(t)$ and $x_L(t)$ are given by

$$x_J(t) = 0 + t + \frac{1}{2} \cdot 3t^2 = t + \frac{3}{2}t^2$$

$$x_L(t) = 10 + 2t + \frac{1}{2} \cdot 2t^2 = 10 + 2t + t^2$$

The amount of time it takes for Jonathan to reach Lilian after she notices can be found by setting $x_J(t) = x_L(t)$ and solving for t.

$$t + \frac{3}{2}t^2 = 10 + 2t + t^2 \Rightarrow t = -3.58\,s, 5.58\,s$$

Since a negative time does not make sense, we pick the positive value of $t = 5.58\,s$. So, Jonathan reaches Lilian in $\boxed{5.58\,s.}$

3. These problems deal with projectile motion!

   (a) We know that the range of a projectile R is given by

   $$R = \frac{v_0^2 \sin 2\theta}{g}$$

   Plugging in the numbers,

   $$R = \frac{2^2 \cdot \sin(2 \cdot 30°)}{9.8} = 0.35\,m$$

   Since $R = 0.35\,m < 10\,m$, $\boxed{\text{the basketball does not reach Andrew.}}$

   Note: To prove the range formula, recall that we can find the total time the ball is in the air t. Then, $R = \Delta x = v_x t = v_0 \cos\theta t$. To find t, notice that the y displacement $\Delta y = 0$. Then, t is

   $$\Delta y = v_y t - \frac{1}{2}gt^2 \Rightarrow 0 = v_0 \sin\theta - \frac{1}{2}gt^2 \Rightarrow t = \frac{2v_0 \sin\theta}{g}$$

   Therefore, we can plug this in to find R.

   $$R = v_0 \cos\theta\Big(\frac{2v_0 \sin\theta}{g}\Big) = \frac{2v_0^2 \sin\theta\cos\theta}{g} = \boxed{\frac{v_0^2 \sin 2\theta}{g}}$$

   (b) Here is the information we are given in the problem:

Knowns	Unknowns
$\Delta x = 4\,m$	$v_0 = ?\,m/s$
$\Delta y = h_{hoop} - h_{Andrew} = 1.5\,m$	
$\theta = 60°$	

x-direction:	y-direction:
$\Delta x = v_x t = v_0 \cos\theta t$	$\Delta y = v_y t - \frac{1}{2}gt^2 = v_0 \sin\theta t - \frac{1}{2}gt^2$
$\therefore x = v_0 \cos\theta t$	$\therefore y = v_0 \sin\theta t - \frac{1}{2}gt^2$

Since we don't know what t is, we can eliminate it by solving the x-direction equation for t and substitute it into the y-direction equation.

$$y = v_0 \sin\theta\left(\frac{x}{v_0\cos\theta}\right) - \frac{1}{2}g\left(\frac{x}{v_0\cos\theta}\right)^2 = x\tan\theta - \frac{g}{2v_0^2\cos^2\theta}x^2$$

Note: This is the equation of the parabola that a projectile follows! If you knew this equation, you could have started here and skipped the derivation.

We can plug in all the numbers and solve for v_0.

$$1.5 = 4\tan(60°) - \frac{9.8}{2 \cdot v_0^2\cos^2(60°)} \cdot 4^2 \Rightarrow v_0 = \boxed{7.6\ m/s}$$

(c) Here is the information we are given in the problem:

Knowns	Unknowns
$\Delta x = 5\ m$	$t = ?\ s$
$v_0 = 10\ m/s$	
$\theta = 10°$	
$t_{reaction} = 0.15\ s$	

Notice that all of the things we know relate to the x-direction. Then, the x-component of v_0 is $v_x = v_0\cos\theta$. Since we have Δx, v_x, and t, we can use $\Delta x = v_x t = v_0\cos\theta t$. Plugging in numbers and solving for t,

$$t = \frac{\Delta x}{v_0\cos\theta} = \frac{5}{10\cos(10°)} = 0.51\ s$$

Since $t = 0.51\ s > t_{reaction} = 0.15\ s$, $\boxed{\text{Andrew dodges in time.}}$

4. We can split this problem into two parts: finding the time it takes for the ball to go up t_{up} and the time it takes for the ball to fall down t_{down}. To find t_{up}, we can list the knowns and unknowns.

Knowns	Unknowns
$\Delta y = 4\ m$	$t_{up} = ?\ s$
$v_0 = 12\ m/s$	
$a = -g$	

We can use $\Delta y = v_0 t + \frac{1}{2}at^2$. In this problem, the equation becomes $\Delta y = v_0 t_{up} - \frac{1}{2}gt_{up}^2$. Plugging in the numbers and solving for t_{up},

$$4 = 12t_{up} - \frac{1}{2} \cdot 9.8t_{up}^2 \Rightarrow t_{up} = 0.4\ s, 2.05\ s$$

We pick $t_{up} = 0.4\,s$ because this corresponds to the ball going straight up from the floor to the ceiling. The longer time of $2.05\,s$ corresponds to the ball overshooting and falling back down to the ceiling! Next, we can make the known unknown table for t_{down}.

Knowns	Unknowns
$\Delta y = -4\,m$	$t_{down} = ?\,s$
$v_0 = -8.1\,m/s$	
$a = -g$	

Notice how everything is negative, since the ball is moving down (you could also make everything positive). Also, notice how $v_0 \neq 12\ m/s$! Since there is gravitational acceleration, the velocity of the ball at the top has changed.

$$v_{top}^2 = v_0^2 - 2g\Delta y = 12^2 - 2 \cdot 9.8 \cdot 4 \Rightarrow v_{top} = 8.1\,m/s$$

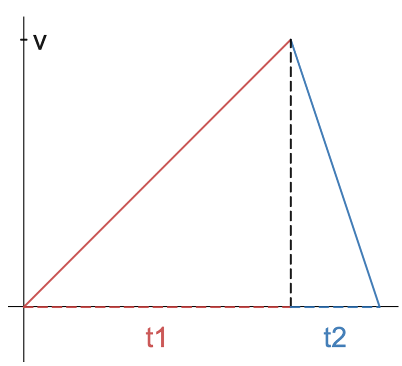
We set $v_{top} = v_0$ when the ball falls from the ceiling to the floor. We have the same variables as before, so we can use $\Delta y = v_0 t + \frac{1}{2}at^2$ again.

$$-4 = -8.09t_{down} - \frac{1}{2} \cdot 9.8t_{down}^2 \Rightarrow t_{down} = -2.05\,s, 0.4\,s$$

We pick $t_{down} = 0.4\,s$ because a negative time does not make sense. So, the total time is $t = t_{up} + t_{down} = \boxed{0.8\,s}$.

Note: We can see that $t_{up} = t_{down} = 0.4\,s$. This is not a coincidence! We just showed that the time it takes to go up and down are equal. If you knew this from the beginning, you could have just multiplied t_{up} by 2 to get the total time.

5. **No calculus solution:** We aren't really given much information in this problem. For example, we don't know how long or far the train accelerates or decelerates for. We can start by drawing a general v-t graph to represent the train's motion.



Let t_1 and t_2 be how long the train accelerates at a_1 and a_2, respectively. We know that the train accelerates to and decelerates from a velocity of $v = a_1t_1 = a_2t_2$. Also, the area under the graph is the distance traveled, so we can write

$$A = \frac{1}{2}bh = \frac{1}{2}(t_1 + t_2)v = s$$

Substituting $t_1 = \frac{v}{a_1}$ and $t_2 = \frac{v}{a_2}$,

$$s = \frac{1}{2}\left(\frac{v}{a_1} + \frac{v}{a_2}\right)v = \frac{v^2}{2}\left(\frac{a_1 + a_2}{a_1a_2}\right) \Rightarrow v = \sqrt{\frac{2sa_1a_2}{a_1 + a_2}}$$

Plugging this back in, we get that t_1 is

$$a_1t_1 = \sqrt{\frac{2sa_1a_2}{a_1 + a_2}} \Rightarrow t_1 = \sqrt{\frac{2sa_2}{a_1(a_1 + a_2)}}$$

Then, we can find that the total time T is given by

$$t_2 = \frac{a_1t_1}{a_2} \Rightarrow T = t_1 + t_2 = t_1\left(1 + \frac{a_1}{a_2}\right) = t_1\left(\frac{a_1 + a_2}{a_2}\right)$$

Plugging in the expression for t_1,

$$T = \left(\frac{a_1 + a_2}{a_2}\right)\sqrt{\frac{2sa_2}{a_1(a_1 + a_2)}} = \boxed{\sqrt{\frac{2s(a_1 + a_2)}{a_1a_2}}}$$

Calculus solution: We don't know how far the train accelerates, so let x and $s - x$ be how far the train accelerates with a_1 and a_2, respectively. Then, using $\Delta x = v_0t + \frac{1}{2}at^2$, we can solve for the times t_1 and t_2.

$$v_0 = 0 \Rightarrow t = \sqrt{\frac{2\Delta x}{a}} \Rightarrow t_1 = \sqrt{\frac{2x}{a_1}},\ t_2 = \sqrt{\frac{2(s - x)}{a_2}}$$

Therefore, we can define a total time function $T(x) = t_1 + t_2$.

$$T(x) = \sqrt{\frac{2x}{a_1}} + \sqrt{\frac{2(s - x)}{a_2}}$$

We want to find a value of x so that $T(x)$ is minimized, since we want the shortest time. So, we can set the derivative to 0 and solve for x.

$$\frac{dT}{dx} = \sqrt{\frac{2}{a_1}}\left(\frac{1}{2\sqrt{x}}\right) - \sqrt{\frac{2}{a_2}}\left(\frac{1}{2\sqrt{s - x}}\right) = 0$$

$$2a_1x = 2a_2(s - x) \Rightarrow x = \frac{sa_2}{a_1 + a_2},\ s - x = \frac{sa_1}{a_1 + a_2}$$

Plugging this value of x into $T(x)$ gives

$$T = \sqrt{\frac{2sa_2}{a_1(a_1+a_2)}} + \sqrt{\frac{2sa_1}{a_2(a_1+a_2)}}$$

Now, all that's left is doing a bit of math to combine the square roots into one to simplify our answer.

$$\begin{aligned} T &= \sqrt{\frac{2sa_2^2}{a_1a_2(a_1+a_2)}} + \sqrt{\frac{2sa_1^2}{a_1a_2(a_1+a_2)}} \\ &= (a_1+a_2)\sqrt{\frac{2s}{a_1a_2(a_1+a_2)}} \\ &= \sqrt{\frac{2s(a_1+a_2)^2}{a_1a_2(a_1+a_2)}} \\ &= \boxed{\sqrt{\frac{2s(a_1+a_2)}{a_1a_2}}} \end{aligned}$$

6. We can split this problem into two parts: before and after the ball bounces off the wall. Before the ball bounces, notice that the ball has to travel a distance of 10 m. Also, the initial and final heights are the same, so we can use the range formula.

$$R = 10 = \frac{v_0^2 \sin 2\theta}{g} \Rightarrow v_0^2 \sin 2\theta = 10g \tag{1}$$

Right now, we have two unknowns (v_0 and θ), so we will need another equation. We can get another equation when we look at what happens after the ball bounces. We know the equation of the parabola a projectile follows is

$$y = x\tan\theta - \frac{g}{2v_0^2\cos^2\theta}x^2$$

Plugging the numbers and the point $(x, y) = (-2, -5)$ in, we get

$$-5 = -2\tan\theta - \frac{4g}{2v_0^2\cos^2\theta} \Rightarrow 5 = 2\tan\theta + \frac{4g}{2v_0^2\cos^2\theta} \tag{2}$$

Now, we have two equations and two unknowns, so we can solve for v_0 and θ. If this system of equations has no solution, then Justin's statement is correct. Otherwise, Justin is wrong. To solve this system, we can first divide by $\sin\theta$ on both sides of (2).

$$\frac{5}{\sin\theta} = \frac{2}{\cos\theta} + \frac{4g}{2v_0^2\cos^2\theta\sin\theta}$$

By (1), we have $v_0^2 \sin 2\theta = 2v_0^2 \sin\theta \cos\theta = 10g$. Substituting this in,

$$\frac{5}{\sin\theta} = \frac{2}{\cos\theta} + \frac{4g}{10g\cos\theta} = \frac{2}{\cos\theta} + \frac{2}{5\cos\theta} = \frac{12}{5\cos\theta}$$

At this point, its looking like we are going to get a solution for this system of equations!

$$\frac{5}{\sin\theta} = \frac{12}{5\cos\theta} \Rightarrow \theta = \tan^{-1}\left(\frac{25}{12}\right) = \boxed{64.4^\circ}$$

Plugging $\theta = 64.4^\circ$ into (1) and solving for v_0,

$$v_0^2 \sin(2 \cdot 64.4^\circ) = 10g \Rightarrow v_0 = \boxed{11.2\,m/s}$$

Therefore, we have concluded that it is possible to make it into the basket, and Justin just needs to get better at the game.