Hyperbolic 3-Manifolds and their Constructions

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Euler Circle

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Euclid's Postulates

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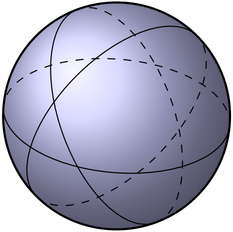

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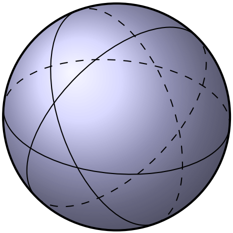

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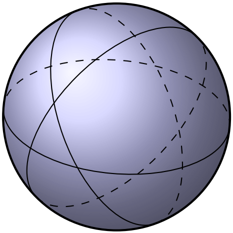

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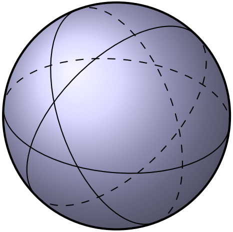

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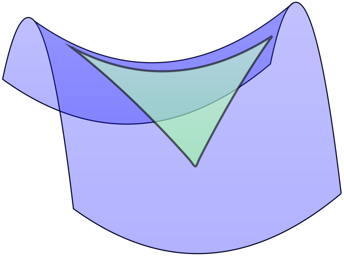

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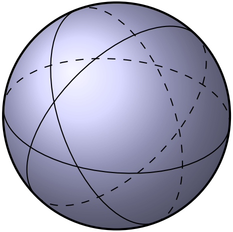

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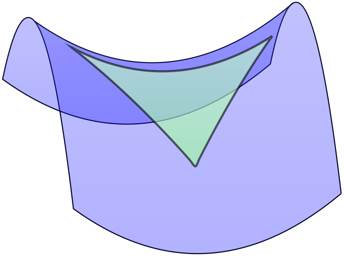

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Hyperbolic n-space

Formal Definitions

Definition 2.1

Euclidean n-space denoted with E^n is an inner product space of $\mathbb{R}^n$ with inner product $\cdot$ such that

$$x \cdot y = x_1 y_1 + \cdots x_n y_n$$

where $x, y \in \mathbb{R}$.

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Definition 2.2

Spherical n-space is

$$S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$$

where $|x| = \sqrt{x \cdot x}$.

Lorentizan n-space

Definition 2.3

Let $x, y \in \mathbb{R}$. The *Lorentizan inner product* is $\circ$ such that

$$x \circ y = x_1 y_1 + x_2 y_2 + \cdots - x_n y_n.$$

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Definition 2.5

Hyperbolic n-space is

$$H^n = \{x \in \mathbb{R}^{n+1} : x_{n+1} > 0 \text{ and } ||x||^2 = -1\}.$$

Hyperboloid Model

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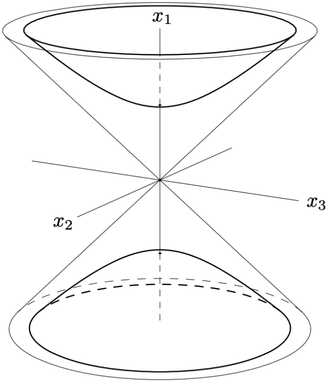


Different Models

Conformal Ball Model

The Map

If

$$B^n = \{x \in E^n : |x| < 1\},$$

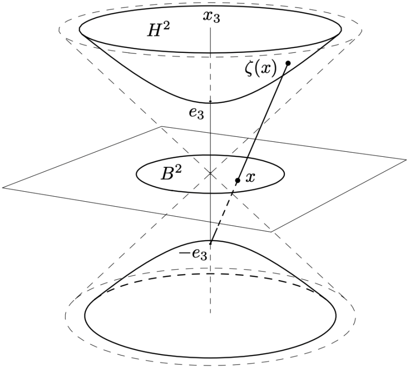
then we consider the *sterographic projection* $\zeta : B^n \to H^n$ defined by

$$\zeta(x) = \left(\frac{2x_1}{1 - |x|^2}, \cdots, \frac{2x_n}{1 - |x|^2}, \frac{1 + |x|^2}{1 - |x|^2} \right)$$

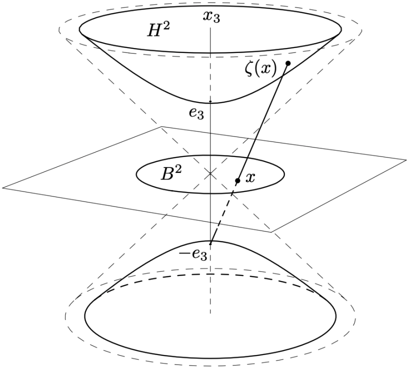
which has an inverse

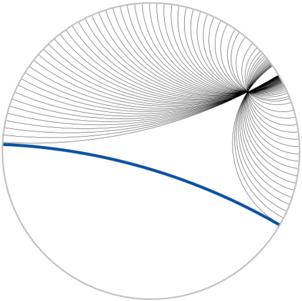
$$\zeta^{-1}(y) = \left(\frac{y_1}{1 + y_{n+1}}, \cdots, \frac{y_n}{1 + y_{n+1}} \right).$$

Conformal Ball Model (contd.)



Conformal Ball Model (contd.)





Projective Disk Model

The Map

If

$$D^n = \{x \in \mathbb{R}^n : |x| < 1\},$$

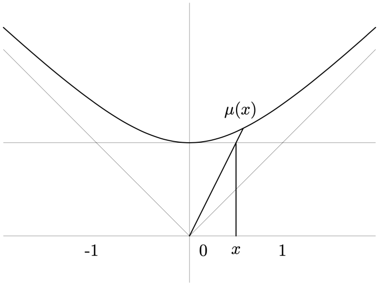
we consider a *gnomonic projection* $\mu : D^n \to H^n$ defined by

$$\mu(x) = \frac{x + e_{n+1}}{|||x + e_{n+1}|||}$$

with an inverse of

$$\mu^{-1}(x) = \left(\frac{x_1}{x_{n+1}}, \cdots, \frac{x_n}{x_{n+1}}\right).$$

Projective Disk Model (contd).



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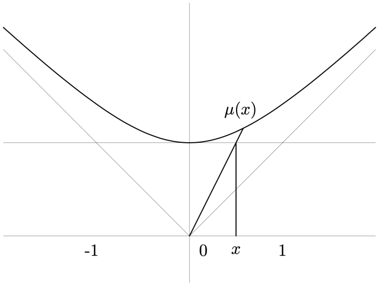

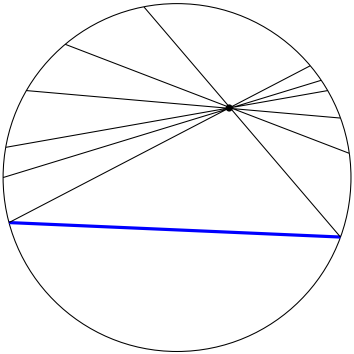

Projective Disk Model (contd.)

Definition 3.1

An m-plane in H^n is the intersection of of H^n with a $(m+1)$ dimensional vector subspace of $\mathbb{R}^{n+1}$ made of vectors with imaginary Lorentizan norms.

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Theorem 3.2

A subset $P \subseteq D^n$ has the property that $\mu(P)$ is a hyperbolic m-plane if and only if P is the nonempty intersection of an m-plane of $\mathbb{R}^n$ and D^n.

Proof of Theorem 3.2

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1. Let Q be an m-plane of H^n and V is the $(m+1)$ dimensional vector space.

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2. Now notice that μ^{-1} is first a radial projection onto the hyperplane L through e_{n+1} and then a vertical translation of $-e_{n+1}$.
3. The radial projection maps Q onto $V \cap L$ so Q maps onto

$$(U \cap C^n) \cap L = U \cap (L \cap C^n) = U \cap (D^n + e_{n+1})$$

where $U \supseteq V$ is an $(m+1)$-plane in $\mathbb{R}^{n+1}$ and C^n is the n dimensional cone.

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4. We translate down and we are done. This process can easily be reversed to convert P into a hyperbolic m-plane.

∎

(X, G)-Manifolds

Manifolds

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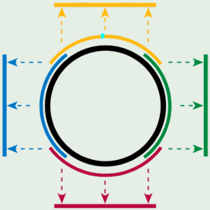

(Wikipedia)

Geometric Spaces

Definition 4.2

For a metric space X, a *geodesic arc* is a distance preserving function $\gamma : [a, b] \to X$. That is,

$$d_1(x, y) = d_2(\gamma(x), \gamma(y))$$

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A *geodesic line* is a locally distance preserving function $\lambda : \mathbb{R} \to X$. That is, for each point $a \in \mathbb{R}$, there is an $r > 0$ such that $x, y \in B_r(a)$ implies that

$$d_1(x, y) = d_2(\gamma(x), \gamma(y))$$

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Geometric Spaces (contd.)

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3. there exists a continuous function $\varepsilon : E^n \to X$ and real $r > 0$ such that ε maps $B_r(0)$ homeomorphically to $B_r(\varepsilon(0))$, and

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Examples

H^n is a geometric space where $\varepsilon(0) = e_{n+1}$ and

$$\varepsilon(x) = (\cosh |x|)e_{n+1} + (\sinh |x|)\frac{x}{|x|} \text{ for } x \neq 0.$$

(X, G)-manifolds

Definition 4.4

Let X be a geometric space, let G be a group of similarities, and let M be an n-manifold. An *(X, G)-atlas* is set of homeomorphisms from open connected subsets of M to open subsets of X

$$\Phi = \{\phi_i : U_i \to X\}$$

such that

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such that

1. The $\{U_i\}$ is an open cover of M and
2. If U_i and U_j overlap, then

$$\phi_j \circ \phi_i^{-1} : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$$

agrees in neighborhood of each point with an element of G.

(X, G)-manifolds (contd.)

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2. A *spherical n-manifold* is $(S^n, \mathrm{I}(S^n))$-manifold, and
3. A *hyperbolic n-manifold* is a $(H^n, \mathrm{I}(H^n))$-manifold.

Gluing Convex Polyhedra

Convex Polyhedra

For this section, $X = E^3$, S^3, or H^3.

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A subset $C \subseteq X$ is called *convex* if for each pair of points $x, y \in C$ such that x and y are distinct and not antipodal when $X = S^3$ there exists a geodesic segment between x and y contained in C.

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Definition 5.2

A *side* of a convex set P is a nonempty, maximal, convex subset of ∂P. If P is nonempty, closed and for each $x \in X$, there is an open neighborhood of x intersecting a finite number of sides of P (or P is *locally finite*), we call P a *convex polyhedron*.

Convex Polyhedra (contd.)

We can also define angles:

Convex Polyhedra (contd.)

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Definition 5.3

Let P be a polyhedron in X and let $x \in P$. The *solid angle* subtended by P at x, is

$$\omega(P, x) = 4\pi \frac{\mathrm{Vol}(P \cap B_r(x))}{\mathrm{Vol}(B_r(x))}$$

where r is less than the distance from x to some side not containing P.

Gluing

Definition 5.4

Let $\mathcal{P}$ be a finite collection of disjoint convex polyhedra in X and let G be a group of isometries of X. A *G-side-pairing* for $\mathcal{P}$ is a subset of G indexed by the set of all sides $\mathcal{S}$ of $\mathcal{P}$

$$\Phi = \{g_S : S \in \mathcal{S}\}$$

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2. the isometries g_S and $g_{S'}$ have the property that $g_{S'} = g_S^{-1}$, and
3. if S is a side of $P \in \mathcal{P}$ and S' is a side of $P' \in \mathcal{P}$, then

$$P \cap g_S(P') = S.$$

Gluing (contd.)

Definition 5.5

Let Φ be a G-side-pairing and let $\Pi = \bigcup_{P \in \mathcal{P}} P$. Two points x and x' in Π are said to be *paired*, notated by $\simeq$, if and only if there is a side S containing x, and x' is in S', and $g_S(x') = x$.

Gluing (contd.)

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Two points x and y in Π are said to be *related*, notated by $\sim$, if and only if $x = y$ or there is a sequence $x_1, x_2, ...x_m$ such that

$$x = x_1 \simeq x_2 \simeq \cdots \simeq x_m = y.$$

Gluing (contd.)

Definition 5.6

The quotient space $\Pi/\sim$ is said to be the space obtained by gluing polyhedra in $\mathcal{P}$ by Φ.

Gluing (contd.)

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Definition 5.7

Let $[x] = \{x_1, x_2, ..., x_n\}$ be a finite equivalence class. Let P_i be the polyhedron in $\mathcal{P}$ that contains x_i. The *solid angle sum* of $[x]$ is

$$\omega[x] = \sum_{i=1}^{n} \omega(P_i, x_i).$$

Gluing (contd.)

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Definition 5.8

A G-side-pairing Φ for $\mathcal{P}$ is *proper* if and only if each equivalence class of Φ is finite and has a solid angle sum of 4π.

Main Theorem

Theorem 5.9

Let G be a group of isometries of X and let M be a space obtained by gluing together a finite collection $\mathcal{P}$ of disjoint convex polyhedra in X by a proper G-side-pairing Φ. Then M is a 3-manifold with an (X, G)-structure.

Thank you!

Thank you everyone, Simon, and Eric.