ENERGY AND MOMENTUM HANDOUT

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1. Energy

1.1. **Work.** In physics, when a constant force F is acting on an particle and displaces it by Δx, work is defined by the equation

$$W = F\Delta x.$$

If the force is not in direction of displacement the definition becomes $W = F\Delta x \cos\theta$. In vector form, this becomes $W = \mathbf{F} \cdot \Delta\mathbf{x}$ where we are taking the dot product. Notice that work is a scalar quantity and its units is called a joule (J).

Now when the force is not constant, in one-dimension, the work is given by

$$W = \int F\,dx.$$

This means that work is the area under the curve of a F vs. x curve.

1.2. **Work Energy Theorem.** The kinetic energy of an particle is defined as

$$K = \frac{1}{2}m|\mathbf{v}|^2$$

where $|\mathbf{v}|$ is the speed of the particle. Notice that this is a scalar quanity like work and also has the same units.

Theorem 1.1 (Work-Energy)**.** *The work done on a particle is equal to the chance in kinetic energy of the particle. That is,*

$$W = \Delta K$$

1.3. **Potential Energy.** A conservative force is defined as a force where the work done to move a particle between two points doesn't depend on the path taken. For example, this is the case for gravity and springs. However, for friction, a longer path will require more work since more energy will be dissipated as heat.

Now in addition to kinetic energy, a particle can also have a potential energy associated with a conservative force defined as

$$\Delta U = -W.$$

This equation basically says that the change in potential energy between two points is negative of the amount of work taken to move the particle from one point to the other. Notice that you can only calculate changes in potential energy so when dealing with potential energy problems, you set one point to have 0 potential energy and calculate the potential energy for any other point.

We can plug in our definition for work to get

$$F = -\frac{\Delta U}{\Delta x} = \frac{dU}{dx}.$$

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This allows you to calculate potential energy from forces. For example, in the case of gravity, we have

$$-mg = -\frac{\Delta U}{\Delta h} \implies \Delta U = mg\Delta h$$

where Δh is the displacement of height. If we set the floor to have 0 potential energy, we get

$$U = mgh$$

where h is the height off the floor. For a spring, if a resting spring has 0 potential energy, we get

$$U = \frac{1}{2}kx^2$$

where x is how much the spring stretches. This are both worthwhile formulas to remember.

1.4. **Conservation of Energy.** Combining the definition for potential energy and work-energy theorem, we get

$$-\Delta U = \Delta K \implies \Delta(U + K) = 0.$$

This is conservation of energy i.e. the sum of the potential and kinetic energy is conserved.

This is the equation used in most energy problems. You write the initial energy and the final energy and set it equal to each other. For example, if the problem involves gravity and a spring, you should write down

$$K_i + U_i^{\text{gravity}} + U_i^{\text{spring}} = K_f + U_f^{\text{gravity}} + U_f^{\text{spring}}.$$

Then you plug in what you know and solve for the unknown.

Now when you deal with non-conservative forces like friction, you have to add an extra term:

$$\Delta(U + K) = W_{\text{nc}}.$$

You can still calculate the initial and final energies like before, but you also have to add the work done by non-conservative forces like friction to the final energy.

2. Momentum

Momentum is defined as

$$\mathbf{p} = m\mathbf{v}$$

which is a vector. We can write a more general version of Newton's second law:

$$\mathbf{F} = \frac{d\mathbf{p}}{dt}.$$

When the mass is constant, this reduces down to $\mathbf{F} = m\,d\mathbf{v}/dt = m\mathbf{a}$.

2.1. **Impulse.** Impulse is defined as

$$\mathbf{J} = \int \mathbf{F}\,dt.$$

Like work, this means that impulse is the area under a force vs. time curve. When the force is constant, this reduces down to $\mathbf{J} = \mathbf{F}\Delta t$. Since $\mathbf{F} = d\mathbf{p}/dt$, we have

$$\mathbf{J} = \Delta\mathbf{p}.$$

This is the equation you will most likely be using. The idea is that a force of a duration of time creates an impulse which changes the momentum of the particle.

2.2. **Conservation of Momentum.** Just like energy, momentum is also conserved. Since $\Delta \mathbf{p} = \mathbf{F}\Delta t$, when there are no external forces, the change in momentum is 0 so momentum is conserved. Just like energy, this is you write the initial and final momentum and set them equal to each other.

The most common place to find conservation of momentum is collision problems. There are two types of collision problems:

(1) Elastic collisions: momentum and kinetic energy are conserved
(2) Inelastic collisions: momentum is conserved but not kinetic energy

Notice that momentum is always conserved but kinetic energy can be lost as heat, sound, etc. In inelastic collisions, there is also completely inelastic collisions where the objects stick together. In general, for collision problems, you write down conservation of momentum and conservation of kinetic energy, if needed, and solve for whatever you need to solve for.

3. Problems

3.1. **Easy.**

(1) Jonathan is dropping his egg drop project off of a $10\,m$ high balcony. How fast is it going just before it hits the ground?
(2) Andrew loves to be carried! Luke wants to get big muscles, so he exercises by lifting Andrew. Andrew weighs $55\,kg$ and Luke lifts him $2\,m$ off the ground every time. If Luke is able to get 8 reps before his arms collapse, how much work does Luke do?
(3) Bob has three identical balls that he wants to throw off a building. He throws ball A at an angle above the horizontal, ball B horizontally, and ball C at an angle below the horizontal. Rank the speeds of the three balls just before they hit the ground.
(4) A $1000\,kg$ rocket ship is moving at $10\,m/s$, which is way too slow! To travel faster, the astronauts turn on the rocket engine, which provides $100\,N$ of thrust. After the rocket moves $10\,m$, how fast is it traveling?
(5) For her physics lab, Chris wants to determine the spring constant of a spring. She does this by holding the spring vertically, dropping a $1\,kg$ potato on it, and measuring how far the spring compresses. She throws the potato down at an initial speed of $2\,m/s$ at a height $1\,m$ above the spring. The spring compresses $0.5\,m$. What is the spring constant?
(6) A roller coaster has a steep drop that immediately goes into a circular loop that is $20\,m$ in diameter. What is the minimum height that the roller coaster needs to drop from so that it can successfully make it around the loop?
(7) Sid is skiing (assume there is no friction) at $5\,m/s$. Unfortunately, he is distracted and doesn't notice that there is a concrete floor ahead, where there is friction. The coefficient of friction $\mu = 0.5$. Sid weighs $10\,kg$.
    (a) How far does he travel before coming to a stop?
    (b) Sid wants to install rocket boosters to his skis so that he can continue traveling at a constant velocity of $5\,m/s$ on the concrete. What is the power required?

3.2. **Hard.**

(1) A bungee-jump cord has length l and is initially folded back on itself. The jumper has mass m, and after she falls off the platform, she is in freefall for a height l. After that, the cord becomes taut and stretches; assume that it acts like an ideal spring

with a particular spring constant. It is observed that the lowest point the jumper reaches is a distance $2l$ below the platform.
   (a) What is the spring constant?
   (b) What is the jumper's acceleration at the lowest point?
   (c) At what position (specify the distance below the platform) is the jumper's speed maximum? What is this speed?
(2) A platform has a rope attached to it which extends vertically upward, over a pulley, and then back down. You stand on the platform. The combined mass of you and the platform is m.
   (a) Some friends standing on the ground grab the other end of the rope and house you up a height h at constant speed. What is the tension in the rope? How much work do your friends do?
   (b) Consider instead the scenario where you grab the other end of the rope and hoist yourself up a height h at constant speed. What is the tension in the rope? How much work do you do?
(3) A block with mass m slides with speed v along a frictionless table toward a stationary block that also has mass m. A massless spring with spring constant k is attached to the second block. The is the maximum distance the spring gets compressed?
(4) A mass m moving with speed v_0 collides elastically with a stationary mass $2m$. Assuming that the final speeds of the masses turn out to be equal, find this speed, and also find the two angles of deflection (relative to the path of the mass m before collision).
   (a) A hose shoots a stream of water vertical upward. The water leaves the hose at speed v and at a mass rate R (kg/s). A horizontal board with mass m is placed a very small distance above the hose and then released. What should m be so that the board hovers at this height? Assume that when the water crashes into the board, it bounces off essentially sideways.
   (b) If you break the board in half, so that its mass is now $m/2$, how high above the hose should it be located if you want it to hover in place?
   (c) In part (a), what should m be if the stream of water is replaced by a stream of marbles that bounce off the board elastically (that is, they bounce off downward with the same speed v)?