Problem Set Solutions for

Differential Topology

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Contents

1 Week 1

(1) We have

$$\begin{aligned} df_{\mathbf{x}}(c_1\mathbf{h}_1 + c_2\mathbf{h}_2) &= \lim_{t\to 0} \frac{f(\mathbf{x} + t(c_1\mathbf{h}_1 + c_2\mathbf{h}_2)) - f(\mathbf{x})}{t} \\ &= \lim_{t\to 0} \frac{f(\mathbf{x} + tc_1\mathbf{h}_1 + t(c_2\mathbf{h}_2)) - f(\mathbf{x} + tc_1\mathbf{h}_1)}{t} \\ &+ \lim_{t\to 0} \frac{f(\mathbf{x} + t(c_1\mathbf{h}_1)) - f(\mathbf{x})}{t} \\ &= \lim_{t\to 0} df_{\mathbf{x}+tc_1\mathbf{h}_1}(c_2\mathbf{h}_2) + df_{\mathbf{x}}(c_1\mathbf{h}_1 \\ &= df_{\mathbf{x}}(c_2\mathbf{h}_2) + df_{\mathbf{x}}(c_1\mathbf{h}_1) \end{aligned}$$

Now notice that

$$\begin{aligned} df_{\mathbf{x}}(a\mathbf{h}) &= \lim_{t\to 0} \frac{f(\mathbf{x} + t(a\mathbf{h})) - f(\mathbf{x})}{t} \\ &= a \lim_{t\to 0} \frac{f(\mathbf{x} + at(\mathbf{h})) - f(\mathbf{x})}{at}. \end{aligned}$$

Setting $at = b$ gives us

$$\begin{aligned} df_{\mathbf{x}}(a\mathbf{h}) &= a \lim_{t\to 0} \frac{f(\mathbf{x} + at(\mathbf{h})) - f(\mathbf{x})}{at} \\ &= a \lim_{t\to 0} \frac{f(\mathbf{x} + b\mathbf{h})}{b} \\ &= a df_{\mathbf{x}}(\mathbf{h}) \end{aligned}$$

Therefore,

$$df_{\mathbf{x}}(c_1\mathbf{h}_1 + c_2\mathbf{h}_2) = df_{\mathbf{x}}(c_1\mathbf{h}_1) + df_{\mathbf{x}}(c_2\mathbf{h}_2) = c_1 df_{\mathbf{x}}(\mathbf{h}_1) + c_2 df_{\mathbf{x}}(\mathbf{h}_2)$$

and we are done.

(2) Let $\phi : V_1 \to U_1$, $\psi : V_2 \to U_2$, $\omega : V_3 \to U_3$ be coordinate charts for X, Y, Z, respectively. First we know that if $f_1 : \mathbb{R}^n \to \mathbb{R}^k$ and $f_2 : \mathbb{R}^k \to \mathbb{R}^m$ are smooth, then $f_2 \circ f_1 : \mathbb{R}^n \to \mathbb{R}^m$ is smooth. This is because if we can take continuous partial derivatives of any order of f_1 and f_2, we can do the same for $f_2 \circ f_1$ by the chain rule.

Next, since f is smooth, we know that $\psi \circ f \circ \phi^{-1}$ is smooth and since g is smooth, we know that $\omega \circ g \circ \psi^{-1}$ is smooth. By the fact from before, we know that

$$\omega \circ g \circ \psi^{-1} \circ \psi \circ f \circ \phi^{-1} = \omega \circ (g \circ f) \circ \phi^{-1}$$

is smooth. Therefore, by definition, $g \circ f$ is smooth.

To prove that $g \circ f$ is a diffeomorphism, we must prove that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ is smooth. Like before, we know that $\phi \circ f^{-1} \circ \psi^{-1}$ and $\psi \circ g^{-1} \circ \omega^{-1}$ are smooth since f^{-1} and g^{-1} are smooth (and this is because f and g are diffeomorphisms). Therefore,

$$\phi \circ f^{-1} \circ \psi^{-1} \circ \psi \circ g^{-1} \circ \omega^{-1} = \phi \circ (f^{-1} \circ g^{-1}) \circ \omega^{-1}$$

is smooth so $f^{-1} \circ g^{-1}$ is also smooth.

(3) Let the dimension of X be $m < n$ and let $\phi : V \to U$ a coordinate chart of X. Additionally, let $f_X : X \to \mathbb{R}$ be the restriction of f to X. Now we know that if $f_X \circ \phi^{-1} : U \to \mathbb{R}$ is smooth, then f_X must smooth but we can easily see that $f_X \circ \phi^{-1}$ must have continuous partial derivatives of all orders since f has continuous partial derivatives of all orders. Therefore $f_X \circ \phi^{-1}$ is smooth so f_X is smooth.

(4)

(a) Consider the map $f : B^n \to \mathbb{R}^n$ defined by

$$f(\mathbf{x}) = \begin{cases} \ln\left(\frac{1+||\mathbf{x}||}{1-||\mathbf{x}||}\right) \cdot \frac{\mathbf{x}}{||\mathbf{x}||} & \mathbf{x} \neq \mathbf{0} \\ \mathbf{0} & \mathbf{x} = \mathbf{0} \end{cases}$$

whose inverse is

$$f^{-1}(\mathbf{x}) = \begin{cases} \frac{1-e^{-||\mathbf{x}||}}{1+e^{-||\mathbf{x}||}} \cdot \frac{\mathbf{x}}{||\mathbf{x}||} & \mathbf{x} \neq \mathbf{0} \\ \mathbf{0} & \mathbf{x} = \mathbf{0} \end{cases}.$$

We can see that both of these are smooth so f is a diffeomorphism.

(b) Let $\phi : V \to U$ be a chart of X. Now consider an open ball B centered at $\phi(x)$. Let $W = \phi^{-1}(B)$ so this is a neighborhood of x. Therefore W is diffeomorphic to B which is diffeomorphic to $\mathbb{R}^n$ (from the previous part) and we are done.

(5) We can clearly see that f is smooth since the derivative is $3x^2$ which is continuous. Now the inverse is $f^{-1}(x) = x^{1/3}$ which has a derivative of $\frac{1}{3}x^{-2/3}$ and this is not defined at $x = 0$ so f is not a diffeomorphism.

(6) First say we pick any point on the set that isn't the origin. We can pick an open interval centered at this point and this is diffeomorphic to itself in $\mathbb{R}^1$. However, if we pick the origin, its neighborhood is T-shaped and which is not diffeomorphic to a neighborhood of any Euclidean space.

(7) The graph of this is either a hyperboloid of one or two sheets or a cone: one sheet when a is positive, two sheets when a is negative, and a cone when $a = 0$. When a is positive, we can split the surface into three parts: the top, middle and bottom. Each of these is diffeomorphic to a neighborhood in $\mathbb{R}^2$ so the X_a is a manifold when $a > 0$. Next, when a is negative, we can easily see that top and bottom sheets are both diffeomorphic to a neighborhood in $\mathbb{R}^2$ so X_a is again a manifold. Finally, when $a = 0$, the problem arises at the origin since there is not neighborhood containing it that is diffeomorphic to a neighborhood in $\mathbb{R}^2$. Therefore X_a is a manifold when $a \neq 0$.

(8) Let x be the point that is removed and consider the plane tangent to the sphere at the point opposite to x. For our diffeomorphism, we can do a stereographic projection where we shoot rays in all directions from x and see where it crosses the sphere and where it hits the plane. However, the plane is $\mathbb{R}^n$ since the sphere lives in $\mathbb{R}^{n+1}$ so S^n with a point missing is diffeomorphic to $\mathbb{R}^n$. Notice that the reason we needed the point to be missing is because the stereographic projection is not defined at x.

(9)

(a) Consider the function

$$f_0(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x = 0 \end{cases}$$

and we claim this is smooth. We know that the two separate pieces are smooth so all we need to do is prove that all their derivatives match up at $x = 0$. The nth derivative of 0 is just 0 while the nth derivative of $e^{-1/x}$ is

$$\frac{d^n}{dx^n}(e^{-1/x}) = \frac{P(x)e^{-1/x}}{x^{2n}}$$

where $P(x)$ is $(n-1)$th degree polynomial. This means that

$$\frac{d^n}{dx^n}(e^{-1/x})\Big|_{x=0} = \lim_{x\to 0} \frac{P(x)e^{-1/x}}{x^{2n}} = 0$$

so $g(x)$ is smooth.

Now we define the function $f(x) = f_0(x-a)f_0(b-x)$ so when $x \notin (a,b)$, either $x-a$ or $b-x$ is negative so either $g(x-a)$ or $f_0(b-x)$ is 0 meaning that $f(x) = 0$. When $x \in (a,b)$, we can clearly see that $f(x)$ is positive so this function satisfies the properties for the problem. We can also easily see that $f(x)$ is smooth since $f_0(x)$ is smooth.

(b) Consider the function

$$g(x) = \frac{\int_a^x f(x)\,dx}{\int_a^b f(x)\,dx}.$$

When $x \in (a,b)$ notice that the numerator is less than the denominator since $f(x) > 0$ so $0 < g(x) < 1$. When $x < a$, we have

$$g(x) = \frac{\int_a^x f(x)\,dx}{\int_a^b f(x)\,dx} = \frac{-\int_x^a f(x)\,dx}{\int_a^b f(x)\,dx} = 0$$

since $f(x) = 0$ when $x < a$. Now when $x > b$, we have

$$g(x) = \frac{\int_a^x f(x)\,dx}{\int_a^b f(x)\,dx} = \frac{\int_a^b f(x)\,dx + \int_x^b f(x)\,dx}{\int_a^b f(x)\,dx} = \frac{\int_a^b f(x)\,dx + 0}{\int_a^b f(x)\,dx} = 1$$

again since $f(x) = 0$ when $x > b$. Finally, we can see that $g(x)$ is smooth since first,

$$\frac{dg}{dx} = \frac{f(x)}{\int_a^b f(x)\,dx}$$

by the Fundamental Theorem of Calculus. Second, higher order derivatives of this are continuous since $f(x)$ is smooth so all of this means that $g(x)$ is smooth.

(c) We can use the previous parts to see that this function is simply $h(\mathbf{x}) = g(||\mathbf{x}||)$ which is smooth.

(11) By definition, we have

$$d\iota_x(v)(\phi) = v(\phi \circ \iota)$$

for any $\phi \in C^\infty(Y)$. Now we can see that $\phi \circ \iota$ is just the restriction of ϕ to $X \subseteq Y$ so $d\iota_x$ must be the inclusion map.

(12) We must find a bijection between T_xU and T_xX and we are already half way done from the previous problem. Since U is a submanifold of X, we know that $d\iota_x : T_xU \to T_xX$ is the inclusion map so it must be injective. All we must prove is that this is surjective. Let $v : C^\infty(X) \to \mathbb{R}$ be a tangent vector in T_xX and let $v' : C^\infty(U) \to \mathbb{R}$ be defined by $v'(\phi \circ \iota) = v(\phi)$ for $\phi \in C^\infty(X)$. Now we prove that this is a tangent vector in T_xU which would mean that $d\iota_x$ is surjective. It suffices to show that v' is a derivation so for any $\phi, \psi \in C^\infty(X)$, we have

$$\begin{aligned} v'((\phi\circ\iota)(\psi\circ\iota)) &= v'((\phi\psi)\circ\iota) \\ &= v(\phi\psi) \\ &= \phi(x)v(\psi) + \psi(x)v(\phi) \\ &= (\phi(x)\circ\iota)v'(\psi\circ\iota) + (\psi\circ\iota)v'(\phi\circ\iota). \end{aligned}$$

(13)

1. We have

$$\begin{aligned} df_x(v+w)(\phi) &= (v+w)(\phi \circ f) \\ &= v(\phi \circ f) + w(\phi \circ f) \\ &= df_x(v)(\phi) + df_x(w)(\phi) \end{aligned}$$

   and

$$\begin{aligned} df_x(cv)(\phi) &= (cv)(\phi \circ f) \\ &= cv(\phi \circ f) \\ &= cdf_x(v)(\phi) \end{aligned}$$

   so df_x is linear.

2. We have

$$\begin{aligned} d(g \circ f)_x(v)(\phi) &= v(\phi \circ (g \circ f)) \\ &= v((\phi \circ g) \circ f) \\ &= df_x(v)(\phi \circ g) \\ &= dg_{f(x)}(df_x(v))(\phi) \\ &= dg_{f(x)} \circ df_x \end{aligned}$$

   so we are done.

(14) Suppose that there indeed exists a diffeomorphism $f : \mathbb{R}^m \to \mathbb{R}^n$. This means that $f \circ f^{-1}$ and $f^{-1} \circ f$ are both identity. By the chain rule, we see that

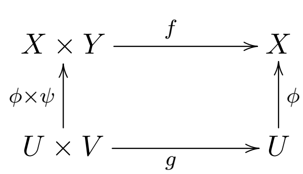
$$df_{\mathbf{x}} \circ df^{-1}_{f(\mathbf{x})} = df^{-1}_{f(\mathbf{x})} \circ df_{\mathbf{x}} = I$$

so $df_{\mathbf{x}}$ is an isomorphism. However, this implies that $\mathbb{R}^m$ and $\mathbb{R}^n$ are isomorphic as vector spaces since $df_{\mathbf{x}}$ is linear but this is clearly a contradiction.

(15)

(a) Let U_1, U_2 be open sets in $\mathbb{R}^{n_1}$, $\mathbb{R}^{n_2}$, respectively, and let $g_1 : U_1 \to \mathbb{R}^{m_1}$ and $g_2 : U_2 \to \mathbb{R}^{m_2}$. Now if $g = g_1 \times g_2 : U_1 \times U_2 \to \mathbb{R}^{m_1} \times \mathbb{R}^{m_2}$ is smooth, Then $dg_{(u_1,u_2)} = d(g_1)_{u_1} \times d(g_2)_{u_2}$. Now we can consider the parameterization $\phi : \widetilde{U_1} \to X$ and $\psi : \widetilde{U_1} \to Y$ where $\widetilde{U_1}$ and $\widetilde{U_2}$ are open sets in $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$ so $dg_{(u_1,u_2)} = d(g_1)_{u_1} \times d(g_2)_{u_2}$ gives a map between the tangent spaces.

(b) We have the following diagram



so $g = \phi^{-1} \circ f \circ (\phi \times \psi)$ is the projection map. This means that $df_{(x,y)} = d\phi_0 \circ dg_{(0,0)} \circ d(\phi \times \psi)_0^{-1}$ is also the projection map.

(16)

(a) We know that $\frac{d\gamma}{dt}(t_0)(\phi) = d\gamma_{t_0}(1)(\phi) = 1(\phi \circ \gamma)$ for some $\phi \in C^\infty(X)$. Now we know that the tangent vector 1 in $\mathbb{R}$ is the same as $1(\psi) = \frac{d\psi}{dt}(t_0) = d\psi_{t_0}(1)$ in $T_{t_0}\mathbb{R}$ for any $\psi \in C^\infty(\mathbb{R})$. This means that

$$\begin{aligned}\frac{d\gamma}{dt}(t_0)(\phi) &= 1(\phi \circ \gamma)\\ &= d(\phi \circ \gamma)_{t_0}(1)\\ &= (d\phi_{x_0} \circ d\gamma_{t_0})(1)\\ &= d\phi_{x_0}((\gamma_1'(t_0), \dots, \gamma_n'(t_0)))\end{aligned}$$

so we can choose a ϕ such that $d\phi_{x_0}$ is identity and we are done.

(b) To go from the velocity vectors to the tangent vectors, since the tangent vectors are directional derivatives, we can just map the velocity vector to the directional derivative in the same direction.

To go from the tangent vectors to the velocity vectors, we can take a tangent vector and project it onto the manifold. Then the velocity vector of this curve must be the same as the tangent vector so all of this means that the tangent space and set of velocity vectors are isomorphic sets.

(17) We have proved the more general case in the next problem since in $\mathbb{R}$, all local diffeomorphisms are injective. The reason the analogous result in $\mathbb{R}^2$ is false is because not all local diffeomorphisms are injective.

(18) Since f is a local diffeomorphism it maps neighborhoods $U \to V$ diffeomorphically. Now consider some open cover $\{U_i\}$ such that $\bigcup_{i=1}^k U_i = X$. This means that

$$f\left(\bigcup_{i=1}^k U_i\right) = \bigcup_{i=1}^k f(U_i)$$

is a diffeomorphism from X to an open subset of Y since the $f(U_i)$'s are open subsets of Y. Now the reason f has to be injective is so that we can take the inverse of the open subset of Y since if we cannot do this, the map we have constructed is not a diffeomorphism.

(19) We must first prove that this is an immersion or that the derivative is injective. The derivative at a is

$$df_a(h) = a\left(\frac{e^h - e^{-h}}{2}, \frac{e^h + e^{-h}}{2}\right)$$

which is an injection. We can also similarly see that the actual immersion is also injective so all that is left to do is prove that the immersion is proper. Now the image of this is just one side of hyperbola which is a 1-manifold with $f(t)$ as a parameterization. This means that all compact sets on the hyperbola map to compact sets in $\mathbb{R}$.

2 Week 2

(1) We know that if $f : X \to Y$ is a submersion with $\dim X = n$ and $\dim Y = m$, then there exists local coordinates around x and y such that $f(x_1, ..., x_n) = (x_1, ..., x_m)$ by the Local Submersion Theorem. Since $n > m$, this is a projection so submersions are local projections. Therefore, since projections map open sets to open sets, the same must be true submersions so we are done.

(2)

(a) From the previous problem since f is a submersion, it must be an open map. Next, since X is compact, since all continuous functions map compact spaces to compact spaces, we know that the image of f is closed. This means that the image is either all of Y or empty since Y is connected. If we assume both manifolds are nonempty, we are done.

(b) Now we know that X is compact and Euclidean space is connected so every submersion is surjective. This means that $f(X) = \mathbb{R}^n$ but this is not true since X is compact and $\mathbb{R}^n$ is not. This is a contradiction so we are done.

(3) We know that the directional derivative $df_a : \mathbb{R}^n \to \mathbb{R}$ at a point (a_1, a_2, a_3) is just

$$df_a(h) = 2a_1h_1 + 2a_2h_2 - 2a_3h_3.$$

Clearly the only point where this is not surjective is (0, 0, 0). This means that 0 is the only critical value and $(0, 0, 0)$ is the only critical point. Therefore for all $a \neq 0$, the set $f^{-1}(a)$ is a submanifold of $\mathbb{R}^3$. We actually proved this in problem 7 of last week's Pset. All these submanifolds are 2-manifolds so for $a, b \neq 0$, we know that $f^{-1}(a)$ and $f^{-1}(b)$ are diffeomorphic when a and b have the same sign.

(4) First since y is regular, by the Local Submersion Theorem, we know that $f^{-1}(y)$ is a submanifold of dimension 0. This is because the dimensions of X and Y are the same. Additionally, since the singleton y is compact, this must mean that $f^{-1}(y)$ is compact so $f^{-1}(y)$ must be finite.

Now we know that f is a local diffeomorphism so there exists neighborhoods $x_i \in V_i'$ and $y \in U_i'$ such that $f : V_i' \to U_i'$ is a diffeomorphism. Now we can shrink the V_i''s such that they are disjoint and we can do this since $f^{-1}(y)$ is finite. Now let $U' = \bigcap_{i=1}^n U_i'$ and let $V_i'' = V_i' \cap f^{-1}(U')$. Clearly $f : V_i'' \to U'$ is a diffeomorphism. Finally, let $Z = f(X \backslash \bigcup V_i'')$ so Z must be closed in Y and it doesn't contain y. Therefore, the sets $U = U' \backslash Z$ and $V_i = V_i'' \cap f^{-1}(U)$ are the sets we are looking for.

(5) We know that complex polynomials are in the form

$$p(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_0$$

so the derivative is

$$dp_{z_0}(w) = c_n n w^{n-1} + c_{n-1}(n-1)w^{n-1} + \cdots + c_1.$$

Since this is a finite polynomial, there is a finite number of points w where this is 0. These clearly are the points where $dp_{z_0}(w)$ is not surjective. This means that these roots are the critical points and 0 is the unique critical value.

(6) Let $M(n)$ be the set of $n \times n$ matrices so $O(n) \subseteq M(n)$. Consider the map $f : M(n) \to M(n)$ defined by $f(A) = AA^T$ so $f^{-1}(I) = O(n)$. Therefore, since f is continuous and I is a closed singleton, we know that $O(n)$ must also be closed. Additionally, for $Q \in O(n)$, we know that $||Q\mathbf{x}|| = ||\mathbf{x}||$ so $O(n)$ is bounded in a 1-ball. Putting all of this together, we have that $O(n)$ is closed and bounded so by Heine-Borel Theorem, we know that $O(n)$ is compact.

(7) To find the tangent space of $O(n)$, we can use the velocity vector definition of the tangent space which we proved is equivalent to the derivation definition of the tangent space in the previous problem set. Now let $\gamma : \mathbb{R} \to O(n)$ be a curve in $O(n)$ such that $\gamma(0) = I$. We also know that for any $t \in \mathbb{R}$, the matrix $\gamma(t)$ must be an orthogonal matrix so

$$\gamma(t)^T \gamma(t) = I.$$

Differentiating both sides with respect to t at $t = 0$ gives us

$$(\gamma(0)^T)'\gamma(0) + \gamma(0)^T \gamma'(0) = 0.$$

Now $\gamma(0) = I$ so
$$(\gamma(0)^T)'I + I\gamma'(0) = 0 \implies (\gamma'(0))^T = -\gamma'(0).$$
Therefore, this means that $T_I O(n) \subseteq \{A \in \mathbb{R}^{n^2} : A^T = -A\}$.

Now we must go the opposite way i.e. we must prove that every skew-symmetric matrix is in the tangent space of $O(n)$. For some A such that $A^T = -A$, consider the curve $\gamma(t) = e^{tA}$. First, this curve goes through I since $\gamma(0) = e^{0A} = I$. Additionally, for any $t \in \mathbb{R}$ we know that

$$(e^{tA})^T e^{tA} = e^{tA^T} e^{tA} = e^{-tA} e^{tA} = I$$

so the curve is indeed in the orthogonal group. Finally,

$$(e^{tA})'|_{t=0} = Ae^{tA}|_{t=0} = A$$

so $A \in T_I O(n)$. This means that

$$\{A \in \mathbb{R}^{n^2} : A^T = -A\} \subseteq T_I O(n) \implies T_I O(n) = \{A \in \mathbb{R}^{n^2} : A^T = -A\}.$$

(8) Consider the map $f : M(n) \to \mathbb{R}$ such that $f(A) = \det(A)$. This means that $f^{-1}(1) = \mathrm{SL}_n(\mathbb{R})$ so we need to determine whether 1 is a regular value or not. The derivative is

$$\begin{aligned} df_A(B) &= \lim_{s\to 0} \frac{f(A + sB) - f(A)}{s} \\ &= \lim_{s\to 0} \frac{\det(A) + s\det(A)\,\mathrm{Tr}(A^{-1}B) + p(A, B) - \det(A)}{s} \\ &= \det(A)\,\mathrm{Tr}(A^{-1}B). \end{aligned}$$

where $p(A, B)$ is a polynomial in s excluding the constant and linear terms so $\lim_{s\to 0} p(A, B)/s = 0$. Now we must determine if this is surjective when $A \in \mathrm{SL}_n(\mathbb{R})$. For matrices that are in this set, the determinant is 1 so $df_A(B) = \mathrm{Tr}(A^{-1}B)$ which we can see is surjective.

To determine the tangent space, let $\gamma : R \to \mathrm{SL}_n(\mathbb{R})$ be a curve such that $\gamma(0) = I$. We know that $\det(\gamma(t)) = 1$. By Jacobi's formula, we have

$$\frac{d}{dt}\det(\gamma(t)) = 0 \implies \det(\gamma(t)) \cdot \mathrm{Tr}\left(\gamma(t)^{-1} \cdot \gamma'(t)\right) = 0.$$

Plugging in $t = 0$ and using $\gamma(0) = I$, we get

$$\mathrm{Tr}(\gamma'(t)) = 0$$

so the tangent space is all matrices that has a trace of 0.

(9) Let A be a $m \times n$ matrix in the form

$$A = \begin{pmatrix} B & C \\ D & E \end{pmatrix}$$

where B is a nonsingular $r \times r$ matrix. This means that the rank of A is at least r and the rank exceeds r if some of the columns of $\begin{pmatrix} C \\ E \end{pmatrix}$ are linearly independent to the columns of $\begin{pmatrix} B \\ D \end{pmatrix}$. We claim that this never happens if $E - DB^{-1}C = 0$. We can see this by multiplying A by the $n \times (n - r)$ matrix

$$\begin{pmatrix} B^{-1}C \\ I \end{pmatrix}.$$

We get

$$\begin{pmatrix} B & C \\ D & E \end{pmatrix} \begin{pmatrix} -B^{-1}C \\ I \end{pmatrix} = \begin{pmatrix} C - BB^{-1}C \\ E - DB^{-1}C \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Now notice that when we multiply by this special matrix, we are taking the linear combination of all the columns of $\begin{pmatrix} B \\ D \end{pmatrix}$ and one of the columns $\begin{pmatrix} C \\ E \end{pmatrix}$ but we are always getting the 0 vector. Therefore, none of the columns of $\begin{pmatrix} C \\ E \end{pmatrix}$ are linearly independent to the columns of $\begin{pmatrix} B \\ D \end{pmatrix}$ so the rank of A is r. Additionally, we can also easily see that the reverse is also true where if a A has rank r, then $E - DB^{-1}C = 0$.

Now let $S(m, n)$ be the set of $m \times n$ matrices and let $f : S(m, n)^4 \to S(m - r, n - r)$ be the function defined as $f(B, C, D, E) = E - DB^{-1}C$. Therefore, we must prove that the 0 matrix is a regular value since $f^{-1}(0)$ is the set that is isomorphic to all rank-r matrices. The derivative is just

$$\begin{aligned} df_{(B_0,C_0,D_0,E_0)}(B, C, D, E) &= E - d(DB^{-1})_{(B_0,C_0,D_0,E_0)}(B, C, D, E)C_0 - D_0B_0^{-1}C \\ &= E - (DB_0^{-1} - D_0B_0^{-1}BB_0^{-1})C_0 - D_0B_0^{-1}C \\ &= E - DB_0^{-1}C_0 + D_0B_0^{-1}BB_0^{-1}C_0 - D_0B_0^{-1}C \end{aligned}$$

which we can see is surjective. Now we can see that the dimension of the submanifold is just $4mn - (m - r)(n - r) = \boxed{3mn + r(m + n) - r^2}$.

(10) First, we choose local parameterizations $\phi : U \to X$ and $\psi : V \to Y$ with $\phi(0) = x$ and $\psi(0) = f(x)$ such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle\phi}\uparrow & & \uparrow{\scriptstyle\psi} \\ U & \xrightarrow[\ g\ ]{} & V \end{array}$$

commutes. In order to use the Inverse Function Theorem, we have decrease g. Since $dg_0 : \mathbb{R}^n \to \mathbb{R}^m$ is surjective, we can choose a basis of $\mathbb{R}^n$ such that it has the form

$$\begin{pmatrix} I_m & 0 \end{pmatrix}$$

where the 0 denotes the $m \times (n - m)$ matrix of all zeros. Now, we define $\widetilde{g}$ as

$$\widetilde{g}(x) = g(x, z) - (0, z)$$

so $\widetilde{g}$ maps an open set of $\mathbb{R}^m$ into $\mathbb{R}^m$, and the matrix of $d\widetilde{g}_0$ is I_m. Therefore by the Inverse Function Theorem, $\widetilde{g}$ is a local diffeomorphism of $\mathbb{R}^m$ at 0 so since ψ and $\widetilde{g}$ are local diffeomorphism at 0, we know that $\psi \circ \widetilde{g}$ is also one. Thus, $\psi \circ \widetilde{g}$ can be used as a local parameterization of Y around $f(x)$. In these coordinates, we see that F has the desired form.

(11) Since $T \pitchfork V$, we know that $\text{im}(dT_x) + T_yV = T_y\mathbb{R}^n$. Since V is a subspace of $\mathbb{R}^n$, we have $T_yV = V$ and $T_y\mathbb{R}_n = \mathbb{R}^n$. Additionally, since T is a linear map, we also see that dT_xT so we get the desired result. We can prove the reverse by using basically the same logic.

Next, we know that $V \pitchfork W$ is the same thing as saying $\iota \pitchfork W$ where ι is the inclusion map. This means that $T_xV + T_xW = T_x\mathbb{R}^n$ for some $x \in V \cap W$ but V, W, and $\mathbb{R}^n$ are already linear vector spaces so the tangent spaces are the same so $V + W = \mathbb{R}^n$. The reverse is also easy to see since we can basically use the same logic.

(12)

(a) Since the xy-plane and z-axis are both subspaces of $\mathbb{R}^3$, they intersect transversally if they span $\mathbb{R}^3$. This is because of the observation in the previous problem. However, it is easy to see that this is true so the answer is $\boxed{\text{Yes}}$.

(b) Now these are two planes that are not parallel so they must span $\mathbb{R}^3$ meaning that they intersect transversally and the answer is $\boxed{\text{Yes}}$.

(c) The plane spanned by $(1, 0, 0)$ and $(2, 1, 0)$ is just the xy-plane so with the y axis, this plane cannot span $\mathbb{R}^3$. Therefore, the answer is $\boxed{\text{No}}$ – the two sets do not intersect transversally.

(d) The spaces don't necessarily intersect transversally because $\mathbb{R}^k \times \{0\}$ and $\{0\} \times \mathbb{R}^m$ could be the x-axis and z-axis in $\mathbb{R}^3$. However, these two axis do not span $\mathbb{R}^3$ so they do not intersect transversally. Therefore, this means that in general, the answer is $\boxed{\text{No}}$. However if k or m is greater than $n/2$, the answer would be yes.

(e) We must see if $V \times \{0\}$ and $\{(x, x)\}$ span $V \times V$. For some $(v_1, v_2) \in V \times V$, where $v_1, v_2 \in V$, we can write it as $(v_1 - v_2, 0) + (v_2, v_2) = (v_1, v_2)$. We see that $(v_1 - v_2, 0) \in V \times \{0\}$ and (v_2, v_2) is in the diagonal so these two sets span $V \times V$. Therefore, they intersect transversally so the answer is $\boxed{\text{Yes}}$. (We can use the result from the previous problem even though V is not Euclidean space because it is isomorphic to Euclidean space.)

(f) We must check to see if the set of symmetric and skew matrices span the set of all $n \times n$ matrices and then we can use the result from the previous problem since $M(n)$ is isomorphic to $\mathbb{R}^{n^2}$. Now for some $n \times n$ matrix A, we can see that $A + A^T$ is symmetric since

$$(A + A^T)^T = A^T + (A^T)^T = A^t + A = A + A^T$$

and $A - A^T$ is skew symmetric since

$$(A - A^T)^T = A^T - (A^T)^T = A^T - A = -(A - A^T).$$

Therefore, we can write A as

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

and this is a linear combination of a symmetric and skew symmetric matrix. This means that these two set of matrices span $M(n)$ so they intersect transversally. Therefore the answer is $\boxed{\text{Yes}}$.

(13) First notice that $X \cap Z \subseteq X$ and $X \cap Z \subseteq Z$ so for some $y \in X \cap Z$, we see that $T_y(X \cap Z) \subseteq T_y(X)$ and $T_y(X \cap Z) \subseteq T_y(Z)$. Therefore,

$$T_y(X \cap Z) \subseteq T_y(X) \cap T_y(Z).$$

Next since $X \pitchfork Z$, we know that

$$\operatorname{codim}(X \cap Z) = \operatorname{codim} X + \operatorname{codim} Z \implies \dim(X \cap Z) = \dim X + \dim Z - \dim Y.$$

This means that

$$\dim(T_y(X \cap Z)) = \dim(T_y(X)) + \dim(T_y(Z)) - \dim Y.$$

However we now that $\dim(U \cap V) = \dim U + \dim V - \dim(U + V)$ where U and V are two vector subspaces. Since $X \pitchfork Z$, we know that

$$\dim(T_y(X) + T_y(Z)) = \dim(T_y(Y)) = \dim(Y).$$

Therefore

$$\begin{aligned}\dim(T_y(X) \cap T_y(Z)) &= \dim(T_y(X)) + \dim(T_y(Z)) - \dim(T_y(X) + T_y(Z)) \\ &= \dim(T_y(X)) + \dim(T_y(Z)) - \dim Y.\end{aligned}$$

Since $\dim(T_y(X \cap Z)) = \dim(T_y(X)) + \dim(T_y(Z)) - \dim Y$ and both sets have the same dimension,

$$T_y(X \cap Z) = T_yX \cap T_yZ.$$

(14) If $f \pitchfork g^{-1}(W)$ then

$$\mathrm{im}(df_x) + T_y(g^{-1}(W)) = T_y(Y).$$

Applying dg_y to both sides gives us

$$dg_y(\mathrm{im}(df_x) + T_y(g^{-1}(W))) = dg_y(T_y(Y)) = \mathrm{im}(dg_y).$$

Simplifying the left hand side gives us

$$dg_y(\mathrm{im}(df_x)) + dg_y(T_y(g^{-1}(W))) = \mathrm{im}(d(g \circ f)_x) + dg_y(T_y(g^{-1}(W))) = \mathrm{im}(dg_y).$$

Since $g \pitchfork W$, we know that $\mathrm{im}(dg_y) + T_z(W) = T_z(Z)$ so adding $T_z(W)$ to both sides gives us

$$\mathrm{im}(d(g \circ f)_x) + dg_y(T_y(g^{-1}(W))) + T_z(W) = T_z(Z).$$

Notice that $dg_y(T_y(g^{-1}(W))) \subseteq T_z(W)$ so $g \circ f \pitchfork W$.

For the reverse, we assume $g \circ f \pitchfork W$ and $g \pitchfork W$. The latter implies that $T_y(g^{-1}(W)) = \mathrm{im}((dg_y)^{-1})$. Now let $w \in T_y(Y)$. Since $dg_y(w) \in T_z(Z)$ and $g \circ f \pitchfork W$, there exists $v \in T_x(X)$ and $u \in T_z(W)$ such that

$$dg_y(w) = dg_y(df_x(v)) + u$$

and $w - df_x(v) \in \mathrm{im}((dg_y)^{-1}) = T_y(g^{-1}(W))$. Therefore, we see that $w = df_x(v) + u'$ where $u' \in T_y(g^{-1}(W))$ which means that

$$T_y(Y) = \mathrm{im}(df_x) + T_y(g^{-1}(W))$$

so $f \pitchfork g^{-1}(W)$ and we are done.

(15) First we see that $f \sim f$ since the homotopy between f and f is the identity map: $H(x,t) = f(x)$ for any $t \in [0,1]$. Next, homotopy is symmetric since if $f \sim g$ where the homotopy between the two functions is H, we can define a homotopy between g and f as $H'(x,t) = H(x, 1-t)$. Finally to prove that homotopy is transitive, let $f_1 \sim f_2$ and $f_2 \sim f_3$. To go from $f_1 to f_3$, we go from f_1 to f_2 in the period $[0, 1/2]$ and from f_2 to f_3 in the period $[1/2, 1]$.

(16) Let $f : Y \to X$. Notice that $f = \mathrm{Id} \circ f$ but the identity map is homotopic to the constant map which means that this is also the case for $\mathrm{Id} \circ f$. Now all maps from Y to X are homotopic to the constant map so they must all be homotopic to each other.

(17) Consider $\mathbb{S}^{2n-1}$. We can think of this as a subset of the complex n-tuple $z = (z_1, z_2, ..., z_n) \in \mathbb{C}^n$ with modulus 1 i.e. $|z| = \sqrt{|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2} = 1$. Now consider the homotopy,

$$H(z,t) = e^{i\pi t}z = (e^{i\pi t}z_1, e^{i\pi t}z_2, ..., e^{i\pi t}z_n).$$

This is clearly smooth and $H(z,0) = z$ and $H(z,1) = -z$.

(18) First, let A be compact. We know that for each x and y in $\mathbb{R}^n$, there exists an M such that $|g(x) - g(y)| < M|x - y|$. This is because the distance between two points can only grow by a finite amount when we apply a smooth function since $g(A)$ is compact. Now let $M_{\max}$ be maximum M over all x and y. Therefore if we want the sum of the volume of the boxes that cover $g(A)$ to be $\leq \varepsilon$, we take a set of boxes that cover A with total volume $\varepsilon/M_{\max}$ (which we can do since A has measure 0) and apply g onto the boxes. Therefore, we have shown that $g(A)$ has measure 0. Now every measure 0 set A can be decomposed into a finite number of compact sets so we are done.

(19) Let $g : \mathbb{Z} \to \mathbb{Q}$ be a bijection. We define $f : \mathbb{R} \to \mathbb{R}$ to be any smooth function such that $f(x) = g(m)$ for all $m \in \mathbb{Z}$ and $x \in (m - 1/3, m + 1/3)$. The critical values of f are dense. Essentially, the function is flat on these intervals so $g(m)$ is a critical value. However, $g(m)$ goes through all the rationals so the set of critical values are the rationals meaning that this set is dense.

(20) We start with $n = 1$. If the matrix is $A = \begin{pmatrix} 0 \end{pmatrix}$, the determinant is $\det(A) = x$ so it has no critical values so the determinant is a Morse function. Now notice that for all $n > 1$, the zero matrix will always be a critical point of the determinant. When $n = 2$, this matrix is the only critical point since for $A = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$, the determinant is $\det(A) = x_1 x_4 - x_2 x_3$. The Hessian matrix evaluated at this critical point is

$$H = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

which is nonsingular meaning that it is invertible. Therefore all critical points are nondegenerate so the determinant is a Morse function. Now for $n > 2$, the Hessian matrix evaluated at the zero matrix becomes the zero matrix which is invertible. This means that the zero matrix is a critical point but degenerate so the determinant is not a Morse function.

3 Week 3

(1) Consider the function that maps the $\partial\mathbb{H}^n$ onto itself through the following composition of functions

$$\partial\mathbb{H}^n \xrightarrow{\phi} \partial X \xrightarrow{f} \partial Y \xrightarrow{\psi^{-1}} \partial\mathbb{H}^n$$

where ϕ and ψ are parameterizations of X and Y respectively. Since the whole function is diffeomorphic and so are ϕ and ψ^{-1}, this implies that f maps ∂X to ∂Y diffeomorphically.

(2) If this was a manifold, then $\partial\mathbb{H}^n$ would map diffeomorphically to square boundary but this is not possible to do without creating corners. These corners make the map not diffeomorphic since derivatives of all orders does not exists at these corners. Therefore, $[0,1] \times [0,1]$ is not a manifold with boundary.

(3) Notice that when $a > 1$, the whole sphere is contained in solid hyperboloid so we can easily see that the intersection is a manifold. However, when $-1 \leq a \leq 1$, the hyperboloid intersects the sphere but these intersection points are not going to be smooth i.e. we won't be able to take any number of derivatives at these points. Therefore, the boundary of the intersection won't be diffeomorphic to $\partial\mathbb{H}^n$ so the intersection is not a manifold. Finally, when $a < -1$, the intersection becomes the empty set which is a manifold with boundary.

(4) No matter how we twist the rectangle, after gluing the two ends together, it is a loop so a neighborhood on the interior of strip can be diffeomorphically mapped to $\mathbb{H}^2 \backslash \partial\mathbb{H}^2$. Additionally, we can see that the boundary of a Möbius strip is a loop or in other words, it's equivalent to a circle. Therefore, the boundary can be mapped to $\partial\mathbb{H}^n$.

(5)

(a) Proving ∂X is closed is the same thing as proving the X° is open. Let $\phi : U \to V$ be a parameterization of X. For every point x in the X°, we know there exists a neighborhood of $\phi^{-1}(x)$ in $\mathbb{H}^n$ that doesn't contain $\partial\mathbb{H}^n$. Let this neighborhood be U. This means that $\phi(U)$ is a neighborhood of x that doesn't contain ∂X so the X° is open.

To see that ∂X is compact, we have already proven that ∂X is closed. Additionally, since X is compact, it is bounded so ∂X must be bounded.

(b) Notice that ∂X can be closed and bounded but this doesn't necessarily mean that X is bounded. For example we can have a manifold in $\mathbb{R}^3$ bounded by a circle but it can extend infinitely in the z direction. Therefore X doesn't need to be compact if ∂X is.

(6) We can pick a boundary point in ∂B^n and just map everything in B^n to this point. Therefore, this map is smooth and the boundary point is a fixed point of the map.

(7) A simple rotation around the center of the torus shows us that there exists smooth functions on the solid torus that don't have any fixed point. The problem happens when we construct the retraction g. We defined $g(x)$ as the point where the ray starting at $f(x)$ through x hits the boundary of B^n. However, in the case of the torus, the ray hits the boundary of the torus many times and this lack of uniqueness breaks the proof.

(8) Now we know that any smooth function can be arbitrarily well-approximated by polynomials which are smooth. Therefore, we can approximate a continuous function as smooth function made up of polynomials. We know that there exists a fixed point in the smooth case so this proves that there exists a fixed point in the continuous case.

(9) We know that the $n \times n$ matrix is the same as a the linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$. Now consider the map $T' : S^n \to S^n$ that is defined by $T'(x) = T(x)/||T(x)||$. We know that S^n is homeomorphic to B^{n-1} so by Brouwer's fixed point theorem, there exists a x_0 such that $T'(x_0) = x_0$. In other words $T(x_0)/||T(x_0)|| = x_0 \implies T(x_0) = ||T(x_0)||x_0$. Therefore x_0 is an eigenvector and $||T(x_0)||$ is the nonnegative eigenvalue.

(10) Consider the distance function from w to any point on Y. This function is clearly continuous and any continuous function on a compact set attains a minimum. Therefore, there exists a closest point y.

Now we must prove that $(w - y) \cdot t = 0$ for every $t \in T_y(Y)$. This is the same as proving that if there exists a $t \in T_{y'}(Y)$ such that $(w - y') \cdot t \neq 0$, then y' is not the closest point to w. Let γ be the curve on Y that passes through y' and is in the same plane as t i.e. t will be tangent to γ. We now shift our perspective to this plane. We know that if there exists a tangent vector t such that $(w - y') \cdot t \neq 0$, then the circle of radius $|w - y'|$ is not tangent to γ since if this was the case, then t would be orthogonal to the circle meaning that it would be orthogonal to $w - y'$ which it clearly is not. This means that there exists some ε such that if we shrink the radius of the circle by ε, then the circle still intersects γ. The intersection points will be closer to w than y' which proves that y' isn't the closes point to w.

(11) We previously showed that for any smooth function $f : X \to \mathbb{R}^n$ and almost every a in some open ball, the function $f_a(x) = f(x) + a$ is transversal to any submanifold of $\mathbb{R}^n$ including Y. If we pick $f = \iota$, the inclusion map, we know that $(\iota(x) + a) \pitchfork Y \implies \iota(x + a) \pitchfork Y \implies (X + a) \pitchfork Y$ for almost every a in some open ball. Therefore, for each open ball, there exists a set of a in the open ball with measure zero where $(X+a) \not\pitchfork Y$. Now $\mathbb{R}^n$ can be formed by a union of countable balls. Each ball has its own measure zero set where $(X+a) \not\pitchfork Y$ so taking the union of these gives us the set of all $a \in \mathbb{R}^n$ where $(X+a) \not\pitchfork Y$ and this union must have measure zero.

(12) Notice the dimension constraint means that X and Z are not transveral when they're intersecting. Next since transversability is generic (which we proved this week), we can deform X by an arbitrarily small amount such that $X \pitchfork Z$. However this means that after this deformation, X will not intersect Z.

Next we prove that the deformation can be kept constant outside some open neighborhood of $X \cap Z$. Since X and Z are both compact, their intersection is also compact. Therefore an open neighborhood of $X \cap Z$ contains points in X outside $X \cap Z$. In other words, if U is the open neighborhood, then $X' = (U \cap X) \backslash (X \cap Z)$ is not empty. Now we deform $U \cap X$ so that it doesn't intersect Z. If we keep the rest of X constant, we are not done yet since this deformation might not be smooth. However, we can move around points in X' to make sure this is true. By the nature of how X' is define, this will still keep X and Z not intersecting.

(13) We first parametrize $N(X)$. Since the dimension of $N(X)$ is 2, we parametrize with two parameters: t and k. Notice that we can parametrize the parabola as $x = (t, t^2)$. This means that the tangent vector is $(1, 2t)$ so all normal vectors for x are $v = k(-2t, 1)$. Therefore

$$h(x, v) = h(t, k) = (t, t^2) + (-2kt, k) = ((1-2k)t, t^2 + k).$$

This means that the Jacobian is

$$J = \begin{pmatrix} 1-2k & -2t \\ 2t & 1 \end{pmatrix}.$$

We want this not to be surjective so since this is a square matrix, we set the determinant to 0:

$$1 - 2k + 4t^2 = 0 \implies k = \frac{4t^2+1}{2}$$

so $(t, (4t^2+1)/2)$ are the critical points. Plugging this into h gives us

$$h\left(t, \frac{4t^2+1}{2}\right) = \left(-4t^3, 3t^2 + \frac{1}{2}\right)$$

and these are the critical values which are the focal points.

(14) We have proved this in Proposition 3.6 when Y is Euclidean space i.e. $Y = \mathbb{R}^n$. Now all we are doing in the normal bundle to Z in Y is that we're approximating Y as Euclidean space with the tangent space $T_z(Y)$. Therefore, locally, notice that $N(Z;Y)$ is the same as $N(Z)$ in $\mathbb{R}^{\dim Y}$. Proposition 3.6, this is why $N(Z;Y)$ is a manifold with dimension $\dim Y$.

(15) First notice that $T_p(\mathbb{S}^n)$ is just the n-plane orthogonal to the line from the origin to p. Therefore $T_p(\mathbb{S}^{n-1})$ is a $(n-1)$-plane orthogonal to the line from the origin to p and is in the $(x_1, ..., x_{n-1})$-plane. Next, $T_p(\mathbb{S}^n)$ is orthogonal to the $(x_1, ..., x_n)$-plane since the line from the origin to p is in the $(x_1, ..., x_n)$-plane. Therefore the vectors orthogonal to $T_p(\mathbb{S}^{n-1})$ in $T_p(\mathbb{S}^n)$ are orthogonal to the $(x_1, ..., x_{n-1})$-plane which means that they are spanned by $(0, ..., 0, 1)$.

(16) Let $\widetilde{U}$ be a open set of Y and let $U = Z \cap \widetilde{U}$. Therefore, there exists a parametrization $\phi : U \times \mathbb{R}^{\operatorname{codim} Z} \to N(Z;Y)$. Notice that $\sigma \circ \phi : U \times \mathbb{R}^{\operatorname{codim} Z} \to U$ sends $(u, v) \mapsto u$ so it is a submersion. Therefore, it follows that σ is also a submersion.

(17) We can define a parametrization for $N(Z;Y)$ essentially in the same way as Proposition 3.6 but instead of considering $\mathbb{R}^M$, we consider Y and $T_z(Y)$. Therefore when we plug in the values of v where $d\phi_y^T v = 0$, we get the parametrization of the embedded Z proving that it is a submanifold.

(18) Both $N(Z;Y)$ and Y are manifolds of the same dimension. Therefore there exists parametrizations $\phi_1 : U_1 \to U_{N(Z;Y)}(Z)$ and $\phi_2 : U_2 \to U_Y(Z)$ where $U_X(W)$ denotes an open neighborhood of W in X. Next, let $\psi : U_1 \to U_2$ be a diffeomorphism. (Open neighborhoods in euclidean space are diffeomorphic to each other since they are all diffeomorphic to the open ball.) All of this means that $\phi_2 \circ \psi \circ \phi_1^{-1}$ is the diffeomorphism from an open neighborhood of Z in $N(Z;Y)$ to an open neighborhood of Z in Y.

(19) These global functions can be like dot products. Therefore $g_i(y) = 0$ can mean that tangent vectors at y are orthogonal to all vectors in direction i in an n-dimensional space. Therefore the condition

$$Z = \{y \in U : g_1(y) = \cdots = g_n(y) = 0\}$$

means that tangent vectors of Z are orthogonal to vectors in all directions so $N(Z;Y)$ is diffeomorphic to $Z \times \mathbb{R}^n$. We can use a similar logic to prove the reverse. If Z has to be orthogonal to vectors in all directions, the above condition must hold.

4 Week 4

(1) Let $p(z) = z^7 + \cos(|z|^2)(1 + 93z^4)$ and let

$$p_t(z) = tp(z) + (1-t)z^7 = z^7 + t(\cos(|z|^2)(1 + 93z^4))$$

be a homotopy. If W is a closed ball large enough, we can see that $p_t(z)$ is never zero on ∂W. This is because

$$\frac{p_t(z)}{z^7} = 1 + t\left(\frac{\cos(|z|^2)(1 + 93z^4)}{z^7}\right)$$

and as $z \to \infty$, the numerator of the expression in the parenthesis grows a lot slower than the denominator so the expression in the parenthesis goes to 0. This means that $\frac{p_t}{|p_t|} : \partial W \to \mathbb{S}^1$ is defined for all t so $\deg_2(\frac{p}{|p|}) = \deg_2(\frac{p_0}{|p_0|})$. However, $p_0 = z^7$ which has a mod 2 degree of 1 so p must have a zero in W°.

(2) Let f_1 be any function that is homotopic to f such that $f_1 \pitchfork g^{-1}(W)$. (If $f \pitchfork g^{-1}(W)$, f_1 could just be f). We know that $I_2(f_1, g^{-1}(W))$ is just the number of points in $f_1^{-1}(g^{-1}(W)) = (f_1^{-1} \circ g^{-1})(W)$. We also know that $g \circ f_1 \pitchfork W$ so $I_2(g \circ f_1, W)$ is the number of points in $(g \circ f_1)^{-1}(W) = (f_1^{-1} \circ g^{-1})(W)$ which proves our claim

(3) Let $i : Y \to Y$ be the identity map. By the definition of contractible, this map is homotopic to the constant map which we call c. Therefore, we have

$$I_2(f, Z) = I_2(f \circ i, Z) = I_2(i, f^{-1}(Z)) = I_2(c, f^{-1})(Z) = I_2(f \circ c, Z) = I_2(c, Z).$$

However, we can always make the constant function not intersect Z by choosing the necessary constant such that this happens so $I_2(c, Z) = 0$ meaning that we are done.

(4) Let X be manifold that is contractible and compact. Let $i : X \to X$ be the identity map and let $Z = \{p\}$ be a closed submanifold of X. By what we proved in the previous problem, we know that $I_2(i, Z) = 0$. However, this is impossible since $i^{-1}(Z)$ clearly contains 1 which is not congruent to 0 modulo 2 so we have our desired contradiction.

(5) We claim that $\mathbb{S}^1$ is not contractible. We know that $\mathbb{S}^1$ is compact so if $\mathbb{S}^1$ is contractible, it must be a point because of the previous problem but we know this isn't true. Therefore the identity map isn't homotopic to a constant map which proves that $\mathbb{S}^1$ is not simply connected.

(6) Let $\tilde{f}$ be a map homotopic to f such that $\tilde{f} \pitchfork Z$. The regular values are $\tilde{f}(X) \cap Z$ so by Sard's Theorem, there exists a $p \notin f(X) \cap Z$. Now let g be the sterographic projection from $\mathbb{S}^n$ to $\mathbb{R}^n$ where p is where the rays start from. By problem 3, we have $I_2(g \circ f, Z) = 0$. Notice that p needs to be outside of $f(X) \cap Z$ such that $(g \circ f)^{-1}(Z)$ and henceforth $I_2(g \circ f, Z)$ is defined. Therefore $I_2(f, Z) = 0$.

(7) In this week's notes, we saw that two circles in $\mathbb{T}^2$ have intersection 1 mod 2. However in $\mathbb{S}^2$, two circles have intersection 0 mod 2 since circles on $\mathbb{S}^2$ are contractible so we can shrink the circles until they don't intersect. Because of this mismatch, we see that $\mathbb{S}^2$ cannot be diffeomorphic to $\mathbb{T}^2$.

(8) Since $\deg_2(f) \neq 0$, we know that $I_2(f, \{y\}) \neq 0$ for all $y \in Y$. This means that $f(X)$ contains y for every $y \in Y$ so $f(X) = Y$. Therefore, we have proved that f is surjective.

(9) For the forward direction, we can let W be the manifold that X traces as it gets deformed into Z. Therefore ∂W is the union of how X looked in the beginning and how it looks in the end proving that W is a cobordism of X and Z. However, a counterexample for the converse is Figure 5. Let X be the top circle and Z be the bottom circles. Figure 5 clearly shows a manifold that is a cobordism of X and Z but it is impossible to deform a circle into two circles so X cannot be deformed into Z.

(10) Since X and Z are cobordant, there exists a manifold $W \subseteq Y \times I$ such that $\partial W = (X \times \{0\}) \cup (Z \times \{1\})$. Now let f be the projection from $Y \times I$ to Y restricted to W. By the Boundary Theorem, we know that $I_2(\partial f, C) = 0$. Next, notice that ∂f is a disjoint union of the inclusions $\iota_X : X \times \{0\} \to X$ and $\iota_Z : Z \times \{1\} \to Z$. Therefore,

$$0 = I_2(\partial f, C) = I_2(\iota_X, C) + I_2(\iota_Y, C) = I_2(X, C) + I_2(Z, C)$$

which means that $I_2(X, C) = I_2(Z, C)$.

(11) In the original statement of Borsuk-Ulam, we considered a function $f : \mathbb{S}^n \to \mathbb{R}^{n+1}$ where $f(-x) = -f(x)$ and we concluded that $W_2(f, 0) = 1$. Now f this is not too different from a function $g : \mathbb{S}^n \to \mathbb{S}^n$ since we can just defined g as $g = f/||f||$ so $g(-x) = -g(x)$ still holds. Finally, we have $W_2(f, 0) = \deg_2(f/||f||) = \deg_2(g)$ which is how we end up with this new statement.

(12) Assume there exists no line through the origin where $p_1, ..., p_n$ vanish. This means that the kernel of the function $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ defined by

$$f(x) = (p_1(x), ..., p_n(x))$$

doesn't contain a line through the origin. Therefore, we can find an n-sphere that doesn't intersect the kernel. This means that there exists a map g from the n-sphere to $\mathbb{R}^{n+1} \backslash \{0\}$ which we can define as

$$g(x) = (p_1(x), ..., p_n(x), 0).$$

Notice that $\text{im}(g)$ doesn't contain 0 since the domain never intersects $\ker(f)$ so $g(x) \neq 0$ on the n-sphere. Additionally, we see that g satisfies the property $g(-x) = -g(x)$ since $p_1, ..., p_n$ are odd functions. Therefore, by Corollary 3.3, the image of g intersects every line containing the origin but this is clearly not true since $\text{im}(g)$ is just a plane. All the lines through the origin not in this plane clearly don't intersect the plane (since the plane doesn't contain the origin) and this is a contradiction. Therefore our original assumption is false and there exists a line in $\mathbb{R}^{n+1}$ through the origin where $p_1, ..., p_n$ vanish.

(13) For every $x \in \mathbb{S}^{n-1}$, define $\tilde{h}(x)$ be the oriented hyperplane throught the origin where its unit normal vector is x. An oriented hyperplane is a hyperplane with the additional information of which side of the hyperplane is positive. We call the side containing x the positive side so $\tilde{h}(-x)$ is the same hyerplane as $\tilde{h}(x)$ but with opposite orientation.

Now by the intermediate value theorem, for every $x \in \mathbb{S}^{n-1}$, we can translate $h(x)$ such that the shifted hyperplane splits C_1 into two pieces of equal volume. Let $P(x)$ be the point the origin gets shifted to after this translation. Additionally, for every $x \in \mathbb{S}^{n-1}$, let $h(x)$ be the oriented hyperplane orthogonal to x that passes through $P(x)$. This means that $h(x)$ splits C_1 in half for every $x \in \mathbb{S}^{n-1}$. Additionally, we still have the property that $h(-x)$ is the same hyperplane as $h(x)$ but with opposite orientation.

We are now ready to define $f : \mathbb{S}^{n-1} \to \mathbb{R}^{n-1}$: let $f(x)$ be the $(n-1)$-tuple where its ith component is the volume of the part of C_{i+1} on the positive side of $h(x)$. By the Borsuk-Ulam theorem, there exists an $x \in \mathbb{S}^{n-1}$ such that $f(x) = f(-x)$. However, notice that the ith component of $f(-x)$ is the volume of C_{i+1} on the negative side of $h(x)$ since $h(x)$ and $h(-x)$ are the same hyperplane but with opposite orientation. This means that $f(x) = f(-x)$ implies that $h(x)$ has split all $C_1, ..., C_n$ into pieces of equal volume so we are done.

(14) Let $d_i : \mathbb{S}^{n-1} \to \mathbb{R}$ be defined as

$$d_i(x) = \inf\{|x - y| : y \in C_i\}.$$

Basically, $d_i(x)$ tells us the distance from x to the closest point in C_i. The reason we need to use an infimum is because C_i could be open. Now we define $f : \mathbb{S}^{n-1} \to \mathbb{R}^{n-1}$ as

$$f(x) = (d_1(x), ..., d_{n-1}(x)).$$

By the Borsuk-Ulam Theorem, there exists an $x \in \mathbb{S}^{n-1}$ such that $f(x) = f(-x)$. If $x \in C_n$, then $f(x)$ and $f(-x)$ have no 0's meaning that $x, -x \in C_n$. If $x \in C_i$ for some $i \neq n$, the ith component of $f(x)$ and $f(-x)$ is 0. If C_i is closed, then $x, -x \in C_i$ so we would be done. If not, then $-x \in \bar{C}_i$ or the closure of C_i. This means that there exists some ε such that $B(x, \varepsilon) \subseteq C_i$ and $B(-x, \varepsilon) \cap C_i \neq \varnothing$. Now if we pick a $y \in B(-x, \varepsilon) \cap C_i$, then $-y \in B(x, \varepsilon) \subseteq C_i$ so $y, -y \in C_i$ and we are done.

(15)

(a) Let j be the total number of jewels and let the string have length j. Instead of each jewel being a point on the string, we consider it as a part of the string of length 1. Therefore, we have divided the string up into j regions representing the j jewels.

Now we define a continuous function function $f : \mathbb{S}^n \to \mathbb{R}^n$. Let $x = (x_1, ..., x_{n+1})$ be a point on $\mathbb{S}^n$. We cut the string n times such that the regions between the cuts have lengths $|x_1|, ..., |x_{n+1}|$. If x_i is positive, then the ith region goes to the first person and if not, this region goes to the second person. This means that in $-x$ we cut in the same places but we reverse who gets what region. Let $f(x)$ be the total length of each type of jewel the first person gets. By the Borsuk-Ulam Theorem, there exists a point $x \in \mathbb{S}^n$ such that $f(x) = f(-x)$. This condition says that even if we reverse who gets what region, the total length of each type of jewel the first person gets stays the same. This means that the cutting and division associated with x is fair. Additionally, since the number of jewels of each type is even, total length of each type of jewel each person gets will be an integer.

(b) Let's say there are 2 types of jewels: red and blue and the are arranged on the string as "rrrbbr." There is no way splitting this evenly in $2 - 1 = 1$ cut. However we can split this evenly with two cuts: "r|rrb|br."

(16)

(a) Notice that we can bend a sphere to form the Alexander horned sphere. First we claim that a torus with a radial slice taken out and ends sealed is homeomorphic to a sphere. This is because we can deform a sphere into a cylinder and then bend it to form this specific version of a torus. Now the Alexander horned sphere is just a bunch of these pseudo tori glued to each other. Notice that the ends never meet so se can keep deforming the ends of the pseudo tori to form more pseudo tori which proves that the Alexander horned sphere is homeomorphic to a normal sphere.

(b) Consider the loop around the top of the Alexander horned sphere where the mess is going on. No matter what we do, we cannot deform this loop into the constant loop so they aren't homotopic. This means that $\mathbb{R}^3 \backslash B$ is not simply connected so we are done.

5 Week 5

(1) Because of the Jordan-Brouwer Separation Theorem, we know that the compact manifold of dimension n embedded in $\mathbb{R}^{n+1}$ splits $\mathbb{R}^n$ into an "inside" and "outside." Therefore we can choose one of these as positive orientation and the other as negative orientation.

(2) First we claim that

$$T_xX \oplus T_xZ = (-1)^{(\dim X)(\dim Z)}(T_xZ \oplus T_xX)$$

for all $x \in X \cap Z$. First T_xX and T_xZ are isomorphic to $\mathbb{R}^{\dim X}$ and $\mathbb{R}^{\dim Z}$, respectively, (since $\dim(T_xX) = \dim X$ and $\dim(T_xZ) = \dim Z$) so we can just prove the claim for these Euclidean spaces. Let $\mathcal{B}_1$ be an ordered basis of $\mathbb{R}^{\dim X}$ and let $\mathcal{B}_2$ be an ordered basis of $\mathbb{R}^{\dim Z}$. This means that since $X \pitchfork Z$ and $\dim X + \dim Z = \dim Y$, we have that $\mathcal{B}_1 \cup \mathcal{B}_2$ is an ordered basis of $\mathbb{R}^{\dim X} \oplus \mathbb{R}^{\dim Z}$ and $\mathcal{B}_2 \cup \mathcal{B}_1$ is an ordered basis of $\mathbb{R}^{\dim Z} \oplus \mathbb{R}^{\dim X}$ so both of these are ordered bases of $\mathbb{R}^{\dim Y}$. Now let T be the unique map that takes $\mathcal{B}_1 \cup \mathcal{B}_2$ to $\mathcal{B}_2 \cup \mathcal{B}_1$ while preserving order. This means that we must prove the sign of $\det(T)$ is $(-1)^{(\dim X)(\dim Z)}$.

To make things easier, we convert the basis vectors that make up $\mathcal{B}_1$ and $\mathcal{B}_2$ to standard basis vectors. First let $\mathcal{B} = \{e_1, ..., e_{\dim X + \dim Z}\}$ be the standard ordered basis of $\mathbb{R}^{\dim X + \dim Z}$. Now let T_1 be order preserving map from $\mathcal{B}_1 \cup \mathcal{B}_2$ to $\mathcal{B}$ and let T_2 be the map such that $T = T_1^{-1} \circ T_2 \circ T_1$ so

$$\det(T) = \det(T_1^{-1}) \det(T_2) \det(T_1) = \frac{1}{\det(T_1)} \det(T_2) \det(T_1) = \det(T_2)$$

so we only need to worry about the sign of $\det(T_2)$.

Notice that T_2 is map that switches the order of the first $\dim X$ with the last $\dim Z$ vectors of $\mathcal{B}$. This means that in matrix form, T_2 is

$$\begin{pmatrix} 0 & I_{\dim X} \\ I_{\dim Z} & 0 \end{pmatrix}$$

where the 0 represents how many ever 0's needed to make the matrix square. In the first row, there is a 1 in the $(\dim Z + 1)$th spot so

$$\det(T_2) = \det \begin{pmatrix} 0 & I_{\dim X} \\ I_{\dim Z} & 0 \end{pmatrix} = (-1)^{\dim Z} \det \begin{pmatrix} 0 & I_{\dim X - 1} \\ I_{\dim Z} & 0 \end{pmatrix}.$$

We can do this $\dim X - 1$ times to get

$$\det(T_2) = (-1)^{(\dim X - 1)(\dim Z)} \det \begin{pmatrix} 0 & I_1 \\ I_{\dim Z} & 0 \end{pmatrix} = (-1)^{(\dim X)(\dim Z)}$$

so we are done proving the claim. Since the orientation of $X \cap Z$ is the orientation of $T_x X \oplus T_x Z$, we have

$$X \cap Z = (-1)^{(\dim X)(\dim Z)} Z \cap X.$$

(3) The

(4) We will first prove that (a) and (b) are equivalent. First if Z is orientable, then we can smoothly choose the orientations of the tangent spaces of points in Z. For each point in Z, we can assign the direction of the normal vector at this point to the orientation of the tangent space. This works since there only two ways the normal vector can point in because $\operatorname{codim}_Y Z = 1$. Now since the orientation of the tangent spaces changes smoothly, this means that the normal vector also smoothly varies.

For the reverse, if the normal vector smoothly varies we can again assign the direction the normal vector is pointing in to the orientation of the tangent space at that point so this orientation will smoothly change. Therefore, we can smoothly assign orientations to the tangent spaces of Z so by definition, we know that Z is orientable.

(5) If the Möbius strip was orientable, then the normal vector would smoothly vary but this is clearly not true since when we move in a loop and come back to our starting position, the normal vector switches direction. Therefore by the previous problem, the Möbius strip is nonorientable.

(6) Assume $X \times Y$ is orientable which means that we can assign orientations to the tangent spaces of $X \times Y$ smoothly. However we know that

$$T_{(x,y)}(X \times Y) = T_x X \times T_y Y$$

so we can assign the orientation of $T_x X$ as whatever the orientation of $T_{(x,y)}(X \times Y)$ and this assignment of orientations will be smooth. Therefore, this means that X is orientable which is a contradiction so $X \times Y$ cannot be orientable.

(8)

(a) Notice that the degree of the reflection map of $\mathbb{S}^n$ is -1. Next, it is also easy to see that $\deg(f \circ g) = \deg(f)\deg(g)$ when $f : X \to Y$ and $g : Y \to Z$. This is because

$$\begin{aligned}
\deg(f \circ g) &= \sum_{x \in (f \circ g)^{-1}(z)} \pm 1 \\
&= \sum_{x \in f^{-1}(y)} \sum_{y \in g^{-1}(z)} \pm 1 \\
&= \left(\sum_{x \in f^{-1}(y)} \pm 1 \right) \left(\sum_{y \in g^{-1}(z)} \pm 1 \right) \\
&= \deg(f)\deg(g).
\end{aligned}$$

Now we can notice that the antipodal map is a composition of $n+1$ reflections since we need to reflect all $n+1$ coordinates so its degree is $(-1)^{n+1}$.

(b) The antipodal map is homotopic to the identity when its degree is 1 which is when n is odd.

(9) Let $z = x + iy$, for $x, y \in \mathbb{R}$, so our equation becomes

$$(x + iy)^2 = e^{-(x^2+y^2)} \implies x^2 - y^2 + 2xyi = e^{-(x^2+y^2)}.$$

This implies that $xy = 0$ so at least one of x and y is 0. Clearly, both cannot be 0 so one or the other is 0. Let us first consider $x = 0$. We have

$$-y^2 = e^{-y^2}.$$

This can never happen for $y \in \mathbb{R}$ since the left is always negative and the right is always positive. When $y = 0$, we have

$$x^2 = e^{-x^2}.$$

This clearly has a solution (which can easily be seen by graphing these functions) so we are done.

(10) The degree of a map is homotopy invariant so if two maps are homotopic, then they have the same degree. For the reverse, let $f, g : \mathbb{S}^1 \to \mathbb{S}^1$ be two functions. We can lift these to the maps $\tilde{f}, \tilde{g} : \mathbb{R} \to \mathbb{R}$ so since the degrees are the same, we have

$$\frac{\tilde{f}(2\pi) - \tilde{f}(0)}{2\pi} = \frac{\tilde{g}(2\pi) - \tilde{g}(0)}{2\pi} \implies \tilde{f}(2\pi) - \tilde{g}(2\pi) = \tilde{f}(0) - \tilde{g}(0).$$

This means that we can construct a straight line homotopy between $\tilde{f}$ and $\tilde{g}$ so f and g are homotopic.

(11) Let g be a function with $\deg(g) = 0$ that is constant on the circle of radius $1/2$. By the previous exercise, we know that g is homotopic to f so we can extend f to the annulus between circles of radius $1/2$ and 1. Now we f is constant on the inner circle so we also extend f to be constant in the inner circle's interior so we are done.

(12) Let $\Delta_{xy} = \{(x, y, x, y) : x \in X, y \in Y\}$ be the diagonal. Additionally, let y_i be an intersection point of Δ_y with itself. This means that $I(\Delta_{xy_i}, \Delta_{xy_i}) = I(\Delta_x, \Delta_x)$. Now if we do this for all y_i, we get

$$I(\Delta_{xy}, \Delta_{xy}) = I(\Delta_x, \Delta_x) \cdot I(\Delta_y, \Delta_y) \implies \chi(X \times Y) = \chi(X)\chi(Y).$$

6 Week 6

(1) First we consider the case when A is orientation preserving. Now every linear isomorphism can be described by where the standard basis vectors go to. Since A is orientation preserving, we see that $\{A(e_1), ..., A(e_n)\}$ is a positively ordered basis of $\mathbb{R}^n$. Now we can define A_t as the linear isomorphism defined by the following positively ordered basis:

$$\{e_1 + (A(e_1) - e_1)t, ..., e_n + (A(e_n) - e_n)t\}.$$

This works since the orientation doesn't change over this homotopy to the vectors as they are transformed over t don't cross over each other.

Now when A doesn't preserve, then let $\tilde{A}$ be A composed with the reflection of the first coordinate. This means that $\tilde{A}$ preserves orientation so it is homotopic through isomorphic maps to the identity map. Therefore, we see that A is homotopic to $(-x_1, x_2, ..., x_n)$.

(2) The statement of this problem is saying that there doesn't exist smooth functions $f, g : \mathbb{S}^2 \times \mathbb{S}^2 \to \mathbb{R}$ such that

$$f(x_1, x_2) = g(x_1, x_2) = 0$$

if and only if $x_1 = x_2$. However, this is clearly true if f and g are both distance functions on $\mathbb{S}^2$ (which are smooth) so either the problem statement is wrong or the definition of globally definable is wrong.

We shall assume the problem statement is correct and we use the following definition of global definability:

Definition. A set Z is globally definable if there exists a neighborhood $Z \subset U$ and smooth submersion $f : U \to \mathbb{R}^2$ such that $Z = f^{-1}(z)$ for some $z \in \mathbb{R}^2$.

Notice that the normal bundle of a point in $\mathbb{R}^2$ is just the set of all vectors in $\mathbb{R}^2$. Now we can pull this back to the normal bundle of Δ:

$$N(\Delta, \mathbb{S}^2 \times \mathbb{S}^2) \cong N(\Delta, U) \cong f^* N(z, \mathbb{R}^2)$$

where the latter is the trivial plane bundle over Δ. However, this clearly isn't true so Δ is not globally definable.

(3) Consider the map $f(x) = 1/2x$ on $\mathbb{R}^k$. In matrix form, this is the $k \times k$ matrix

$$\begin{pmatrix} 1/2 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1/2 \end{pmatrix}.$$

This happens to be the same matrix for df_x so $\det(df_x - I) = (-1/2)^k$ so $L_0(f) = (-1)^k$.

Now if we use the same construction as $\mathbb{S}^2$, we will still have fixed points at the north and south poles. However, the source at the north pole has a local Lefschetz number of $L_N(f) = (-1)^n$ because of our reasoning from before. However $L_S(f)$ still remains 1 so $L(f) = 1 + (-1)^n$.

(4) Notice this map is not Lefschetz so we must use the second definition of the local Lefschetz number. We can let B be the circle of radius 1 centered at the origin. We have

$$F(z) = \frac{z + z^m - z}{|z + z^m - z|} = \frac{z^m}{|z^m|} = e^{i(m\theta)}$$

where $z = e^{i\theta}$. This means that we can define the isomorphic function $\tilde{F} : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$F(r, \theta) = (1, m\theta).$$

This wraps around the original circle m times so $L_0(f) = m$.

(5) Let z be a fixed point of z. The derivative is

$$\begin{aligned} df_z(h) &= \lim_{t \to 0} \frac{f(z+th) - f(z)}{t} \\ &= \lim_{t \to 0} \frac{z + th + (z+th)^m + c - (z + z^m + c)}{t} \\ &= \lim_{t \to 0} \frac{th + mz^{m-1}th + \binom{m}{2} z^{m-2}(th)^2 + \cdots}{t} \\ &= h + mz^{m-1}th = h(1 + mz^{m-1}) \end{aligned}$$

so fixed points satisfy $h = h(1 + mz^{m-1})$. However, since $m > 0$, we know that $1 + mz^{m-1} \neq 0$ so the derivative doesn't have any nonzero fixed points. Therefore, this proves that f is Lefschetz.

Now the fixed points of f satisfy $z^m = -c$ so they are the mth roots of unity scaled by $\sqrt[m]{-c}$. When $m = 1$, the only fixed point is $-c$ so its Lefschetz number is $L_{-c}(f) = 1$ since $f = 2z + c$ which is a source at $-c$. When $m > 1$, to find the local Lefschetz numbers of these fixed points, we must determine the sign of $\det(df_z - I)$. Notice that $df_z - I = hmz^{m-1}$ so this map rotates and scales so the corresponding matrix is of the form

$$\lambda \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Therefore, we know that $L_z(f) = 1$ so this means that for all m, the Lefschetz number is $L_z(f) = 1$.

(6) This map is not Lefschetz so we must use the second definition of the local Lefschetz number. Let B be the circle of radius 1 centered at the origin. We have

$$F(z) = \frac{z + \bar{z}^m - z}{|z + \bar{z}^m - z|} = \frac{\bar{z}^m}{|\bar{z}^m|} = e^{i(-m\theta)}$$

where $z = e^{i\theta}$. This means that we can define the isomorphic function $\tilde{F} : \mathbb{R}^2 \to \mathbb{R}^2$ as

$$F(x, y) = (\cos(-m\theta), \sin(-m\theta)) = (\cos(m\theta), -\sin(m\theta)).$$

This wraps around the original circle m times in the negative direction so $L_0(f) = -m$.

(7) Consider the function $f : O(n) \to O(n)$ defined by $f(Q) = AQ$ where $A \in O(n)$ and $A \neq I$. This works since $(AQ)^T AQ = Q^T A^T AQ = Q^T Q = I$. Now this map doesn't have any fixed points since if Q was a fixed point, then $Q = AQ$ but Q is invertible (since it's orthogonal) so $A = I$ which is a contradiction. This means that $L(f) = 0$. Finally, notice that f is homotopic to the identity map by the straight line homotopy:

$$f_t = Q + (AQ - Q)t.$$

Therefore, we know that $L(\mathrm{Id}) = \chi(O(n)) = 0$.

(8) The general method is to define the function $f : X \to X$ as the multiplication of an element by another element (that isn't identity) like in the case of the orthogonal group. This works since group multiplication is closed. Now we can see that this map doesn't have any fixed points like the previous problem and we can see this by multiplying both sides by the inverse. Therefore, this means that $L(f) = L(\mathrm{Id}) = 0$ since f is homotopic to the identity map by the straight line homotopy.

7 Week 7

(1) I don't know how to draw it in Tikz but I can describe it. On the x-axis, the vectors point to the left while on the y-axis, the vectors point to the right. On the $y = x$ line, the vectors point up while on the $y = -x$ line, the vectors point down. The rest of the vectors are defined to make the assignment of tangent vectors smooth. This means that when we go from the positive x-axis to the negative one, we rotate clockwise once and the same when we come back to the positive x-axis so the index is -2.

(2) Notice that the curve $t \mapsto h_t(z)$ is just a line through z and the origin. However, we can see that $\mathbf{v}$ is a source meaning that all the vectors point out from the origin. This means that our line is tangent to $\mathbf{v}$ so it is the flow of $\mathbf{v}$.

Now we can see that the zero at the origin has index 1 since $\mathbf{v}$ is a source so $\mathrm{ind}_0(\mathbf{v}) = 1$. Additionally, we can see that $L_0(h_t) = 1$ so $\mathrm{ind}_0(\mathbf{v}) = L_0$.

(3) We can see that $\mathbf{v}$ is a circulation in the counterclockwise direction so $\mathrm{ind}_0(\mathbf{v}) = 1$. Additionally, just like before $L_0(h_t) = 1$ so $\mathrm{ind}_0(\mathbf{v}) = L_0$.

(4) We know that a torus is just a circle rotate along another circle. Let S be the circle that is actually rotating. At each point on S, we draw the tangent vector pointing to the right and this is our nonvanishing vector field. Specifically, the vector field is $\mathbf{v}(x, y) = (y, -x)$ Now as we rotate S, we draw this vector field and when we are done, we have a nonvanishing vector field for the torus.

(5) We can imagine taking the surface and holding it vertically. Then we pour water at the top and track how the particles fall. Pouring water acts like a source and the path of the particles determine the vector field with 1 zero.

(6) We claim that the vector field $\mathbf{v}(x_1, x_2, ..., x_{n+1}) = (x_2, -x_1, x_4, -x_3, ..., x_{n+1}, -x_n)$ where $(x_i, x_{2i}) \to (x_{2i}, -x_i)$ for $i = 1, 2, ..., (n+1)/2$. This definition works since n is odd so $n+1$ is even.

First, we must check that this is a valid vector field on $\mathbb{S}^n$ i.e. we must check that all the vectors are tangent to $\mathbb{S}^n$. This is indeed the case since

$$(x_1, x_2, ..., x_{n+1}) \cdot (x_2, -x_1, x_4, -x_3, ..., x_{n+1}, -x_n) = \mathbf{0}.$$

Next, the vector field is nowhere vanishing since the only place it vanishes is when $x_1 = \cdots = x_{n+1} = 0$ but this is never the case on $\mathbb{S}^n$.

(7) When $X = \mathbb{R}^n$, we see that $T_x(X) = T_x(\mathbb{R}^n) = \mathbb{R}^n$. This means that $df_x : \mathbb{R}^n \to R$ which is defined by

$$df_x(h) = \lim_{t \to 0} \frac{f(x + th) - f(x)}{t}.$$

This means that

$$\lim_{t \to 0} \frac{f(x + tw) - f(x)}{t} = \nabla(f) \cdot w.$$

The left hand side is just the directional derivative so since we are dealing with Euclidean space,

$$\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right) \cdot w = \nabla(f) \cdot w$$

which implies

$$\nabla(f) = \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right).$$

(8) The zeros of the gradient field is all x such that $\nabla(f)(x) = 0$. This can be written as $\nabla(f)(x) \cdot w = 0$ for any $w \in T_x(X) = \mathbb{R}^n$. By the definition of the gradient field, we have $df_x(w) = 0$ which is the definition of a critical point at x.

Next, if x is a nondegenerate zero of $\nabla(f)(x)$, then the map $d(\nabla(f))_x : T_x(X) \to T_x(x)$ is an bijection. This happens if and only if $d^2 f_x \neq 0$ so x is a nondegenerate critical point of f. Finally, all nondegenerate zeros are isolated zeros so if we pick an f such that all critical points of f are nondegenerate, we see that $\nabla(f)$ is our desired vector field.

(9) The sum of the indices of $\mathbf{v}$ at its zeros inside W is just $\chi(W \backslash \partial W)$. (What is the degree of a map?)

(10) We know that $\chi(X) = V - E + F$. Therefore,

$$6\chi(X) = 6V - 6E + 6F = \left(\sum_{v \in V} 6 \right) - 6E + 6F.$$

Now notice that each face as 3 edges giving us a total of $3F$ edges which is an over count since we are double counting the edges. (An edge will be counted by the two faces it connects.) This means that $E = 3F/2$ so

$$6\chi(X) = \left(\sum_{v \in V} 6 \right) - 6 \cdot \frac{3F}{2} + 6F = \left(\sum_{v \in V} 6 \right) + 3F.$$

We can count the number of faces using $\sum_{v\in V}\kappa(v)$ but this is an overcount since each face has 3 vertices and we are triple counting. Therefore

$$6\chi(X) = \left(\sum_{v\in V} 6\right) + 3\cdot\frac{1}{3}\sum_{v\in V}\kappa(v) = \sum_{v\in V}(6-\kappa(v)).$$

(11) Tiling a sphere with hexagons and pentagons is still triangulation since a hexagon is six triangles and a pentagon is five. This means that $\chi(X) = V - E + F = 2$. Let's say there are n hexagons and m pentagons. First we count the number of vertices. Each shape has one vertex inside it when we split it into triangles so this is a total of $n+m$ inside vertices. Next, we assume that 3 facets meet at one vertex so the total number of outside vertices are triple counting giving is $(6n+5m)/3$. This means that $V = (9n+8m)/3$.

Next we go onto edges. First there are a total of $6n+5m$ edges inside the shapes. Each outside edge is double counted giving us a total of $(6n+5m)/2$. Adding these up gives us $E = 3(6n+5m)/2$. Finally, the hexagon has six faces and the pentagon has five so $F = 6n+5m$. Putting everything together gives us

$$\frac{9n+8m}{3} - \frac{3(6n+5m)}{2} + 6n + 5m = 2.$$

Simplifying the left hand side gives us

$$\frac{9n+8m}{3} - \frac{6n+5m}{2} = \frac{m}{6} = 2 \implies m = 12$$

which gives us the constraint that there must be 12 pentagons. This also shows that we should be able to tile the sphere with 12 pentagons and n hexagons for $n \neq 1$.

(12) Let X be our lie group and let $f_h : X \to X$ be defined by $f_h(x) = h\cdot x$ where $h \in X$ and $\cdot$ is the group operation for X. Because of closure, notice that f_h is a bijection and we can also see that it is differentiable. Now we can define our vector field $\mathbf{v} : X \to \mathbb{R}^n$ as

$$\mathbf{v}(h) = d(f_h)_e(\mathbf{v}(e))$$

where e is the identity element. It is easy to see that this vector field never vanishes if $\mathbf{v}(e) \neq 0$ since $df_h \neq 0$.

8 Week 8

(1) We see that

$$\frac{\partial f}{\partial x} = 3x^2 - 3y^2 \text{ and } \frac{\partial f}{\partial y} = -6xy.$$

We need these to both be 0. From the second partial derivative, this means that at least one of x or y need to be 0. When either x or y become 0, the only way to make the first partial derivative 0 is by making the other variable 0. Therefore the only critical point is $(0,0)$. The Hessian at this point is

$$\begin{pmatrix} 6x & -6y \\ -6y & -6x \end{pmatrix}\Bigg|_{(0,0)} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

which is clearly singular. Therefore $(0,0)$ is degenerate.

(2) We first find the critical points of of this new function. Notice that $d(f+g)_x = f'(x)$ and $d(f+g)_y = g'(x)$. Therefore, if (x,y) is a critical point, then $f'(x) = g'(x) = 0$. Therefore, to get the set of critical points of $f+g$, we take the Cartesian product of the set of critical points of f and g. Since the critical points of f and g are nondegenerate, the same must be true for $f+g$ so its a Morse function.

(3) If $a = x + v$ is a critical value of E, we know that $(x, v) \in \mathcal{N}$ is a critical point of E. Let $n : \widetilde{U} \to \mathcal{N}$ be the parameterization that sends the first n coordinates to $\tilde{x} \in X$ and the next $N - n$ coordinates to $\tilde{v} \in \mathbb{R}^N$ such that $v \perp T_{\tilde{x}}X$. This means that the Jacobian of E at (x, v) is

$$\begin{pmatrix} \frac{\partial x}{\partial u_1} & \cdots & \frac{\partial x}{\partial u_n} & \frac{\partial x}{\partial u_{n+1}} & \cdots & \frac{\partial x}{\partial u_N} \end{pmatrix}$$

and this is singular. However, notice that this matrix is singular if and only if the Hessian matrix at a,

$$h_{ij} = \left(\frac{\partial x}{\partial u_i} \cdot \frac{\partial x}{u_j} - v \cdot \frac{\partial^2 x}{\partial u_i \partial u_j} \right),$$

is singular so we are done.

(4) We first show that this is a well-defined function. That is, we must prove that the same point doesn't get mapped to different real numbers. Two points $x, y \in \mathbb{CP}^n$ are equal if $x = \lambda y$ for some $\lambda \in \mathbb{C}\backslash\{0\}$. We have

$$f(\lambda z_0, ..., \lambda z_n) = \frac{\sum_{j=0}^n j|\lambda z_j|^2}{\sum_{j=0}^n |\lambda z_j|^2} = \frac{(n+1)|\lambda|^2 \sum_{j=0}^n j|z_j|^2}{(n+1)|\lambda|^2 \sum_{j=0}^n |z_j|^2} = f(z_0, ..., z_n).$$

If we let $z_j = x_j + iy_j$, we can turn f into a real valued function:

$$f(x_0, ..., x_n, y_0, ..., y_n) = \frac{\sum_{k=0}^n k(x_k^2 + y_k^2)}{\sum_{k=0}^n x_k^2 + y_k^2}.$$

This means that

$$\frac{\partial f}{\partial x_j} = \frac{2x_j \sum_{k=0}^n (j-k)(x_k^2 + y_k^2)}{\left(\sum_{k=0}^n x_k^2 + y_k^2\right)^2}$$

and

$$\frac{\partial f}{\partial y_j} = \frac{2y_j \sum_{k=0}^n (j-k)(x_k^2 + y_k^2)}{\left(\sum_{k=0}^n x_k^2 + y_k^2\right)^2}.$$

Now we set these derivatives to 0. Since all x_k and y_k cannot be 0, we can clear the denominator which gives us

$$2x_j \sum_{k=0}^n (j-k)(x_k^2 + y_k^2) = 0,$$

$$2y_j \sum_{k=0}^n (j-k)(x_k^2 + y_k^2) = 0$$

for all $0 \le j \le n$. Multiplying the bottom equation by i and adding the two equations gives us

$$2z_j \sum_{k=0}^n (j-k)|z_k|^2 = 0.$$

This means that the only way to make this equation true for all j is to make z_j is nonzero, for some j, and make the rest of the z_k's zero. Therefore the critical points are

$$\{(1, 0, ...0), (0, 1, ..., 0), ..., (0, 0, ..., 1)\}.$$

Now we must make sure the hessian at these points is not singular. Let us say that we are considering the critical point where $z_j \neq 0$. Now notice that so far, we have been working with coordinates of $\mathbb{C}^{n+1}$ and not $\mathbb{CP}^n$ since our coordinates are not scaling invarient. When we compute the Hessian, we must make sure we are working with coordinates of $\mathbb{CP}^n$ so in order to prevent z_j from scaling, we plug in $x_j = 1$ and $y_j = 0$ into our function and remove x_j and y_j as coordinates:

$$f = \frac{j + \sum_{k \neq j} k(x_k^2 + y_k^2)}{1 + \sum_{k \neq j} x_k^2 + y_k^2}$$

and now we can take second order derivatives.

We can see that the only second order derivatives that are nonzero are

$$\frac{\partial^2 f}{\partial x_k^2}, \frac{\partial f}{\partial x_k y_k}, \frac{\partial f}{\partial y_k x_k}, \frac{\partial^2 f}{\partial y_k^2}$$

so the Hessian is just a 2×2 grid of $n \times n$ diagonal matricies. Therefore, the Hessian is nonsingular at the critical points. We can also find the index of the critical point where $z_j \neq 0$ is $2j$.

(5) First we must check if f is invarient to scaling to make sure it is well defined:

$$f(cx, cy, cz) = \frac{(cy)^2 + 2(cz)^2}{(cx)^2 + (cy)^2 + (cz)^2} = \frac{c^2(y^2 + 2z^2)}{c^2(x^2 + y^2 + z^2)} = \frac{y^2 + 2z^2}{x^2 + y^2 + z^2}.$$

Next we find this function's critical points. Our partial derivatives are

$$\frac{\partial f}{\partial x} = \frac{-2x(y^2 + 2z^2)}{(x^2 + y^2 + z^2)^2}$$

$$\frac{\partial f}{\partial y} = \frac{2y(x^2 + y^2 + z^2) - 2y(y^2 + 2z^2)}{(x^2 + y^2 + z^2)^2} = \frac{2y(x^2 - z^2)}{(x^2 + y^2 + z^2)^2}$$

$$\frac{\partial f}{\partial z} = \frac{4z(x^2 + y^2 + z^2) - 2z(y^2 + 2z^2)}{(x^2 + y^2 + z^2)^2} = \frac{2z(x^2 + y^2)}{(x^2 + y^2 + z^2)^2}$$

which means that the critical points are $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ with indices 0, 1, 2, respectively.

(6)

(a) Let $f : \mathbb{CP}^3 \to \hat{\mathbb{C}}$ be defined by $f(z_0, z_1, z_2, z_3) = (z_0^2 + z_1^2 + z_2^2 + z_3^2)/z_0^2$ wehre $\hat{\mathbb{C}}$ is the extend complex plane that includes ∞. Therefore, we can let $z_0 = 0$. The derivative is

$$\begin{aligned} df_a(h) &= \frac{a_0^2(2a_0h_0 + 2a_1h_1 + 2a_2h_2 - 2a_3h_3) + 2a_0h_0(a_0^2 + a_1^2 + a_2^2 + a_3^2)}{a_0^4} \\ &= \frac{a_0(2a_0h_0 + 2a_1h_1 + 2a_2h_2 + 2a_3h_3) - 2h_0(a_0^2 + a_1^2 + a_2^2 + a_3^2)}{a_0^3} \\ &= \frac{(2a_1h_1 + 2a_2h_2 + 2a_3h_3) - (a_1^2 + a_2^2 + a_3^2)}{a_0^3} \end{aligned}$$

which is surjective for all $a \in \mathbb{CP}^3$ except $a_1 = a_2 = a_3 = 0$ which means that 1 is the only critical value. By the Preimage theorem, since 0 is a regular value, we see that $Q = f^{-1}(0)$ is a manifold of dimension $6 - 2 = 4$.

(b) Let $u = x + iy$ where $|u| = \sqrt{x^2+y^2} = 1 \implies x^2+y^2 = 1$. This means that when we multiply by u,

$$x_j \to xx_j - yy_j$$
$$y_j \to xy_j + x_j y.$$

Therefore, we have

$$\begin{aligned}
\widetilde{f}(uz) &= \lambda((xx_0 - yy_0)(xy_1 + x_1 y) - (xx_1 - yy_1)(xy_0 + x_0 y)) \\
&+ \mu((xx_2 - yy_2)(xy_3 + x_3 y) - (xx_3 - yy_3)(xy_2 + x_0 2)) \\
&= \lambda(-x_1 x^2 y_0 + x_0 x^2 y_1 - x_1 y^2 y_0 + x_0 y^2 y_1) \\
&+ \mu(-x_3 x^2 y_2 + x_2 x^2 y_3 - x_3 y^2 y_2 + x_2 y^2 y_3) \\
&= \lambda(x^2+y^2)(x_0 y_1 - x_1 y_0) + \mu(x^2+y^2)(x_2 y_3 - x_3 y_2) \\
&= \lambda(x_0 y_1 - x_1 y_0) + \mu(x_2 y_3 - x_3 y_2) \\
&= \widetilde{f}(z).
\end{aligned}$$

(c) Let $f(z) = \widetilde{f}(z/|z|)$ for $z \neq 0$. We know that $f : \mathbb{CP}^3 \to \mathbb{R}$ since first, 0 is not included in $\mathbb{CP}^3$ and second, for $a \in \mathbb{C}\backslash\{0\}$, we have

$$f(az) = \widetilde{f}\left(\frac{az}{|az|}\right) = \widetilde{f}\left(\frac{a}{|a|} \cdot \frac{z}{|z|}\right).$$

Since $a/|a|$ has magnitude 1, we get

$$f(az) = \widetilde{f}\left(\frac{a}{|a|} \cdot \frac{z}{|z|}\right) = \widetilde{f}\left(\frac{z}{|z|}\right) = f(z).$$

(7)

(a) Without loss of generality, since either z_0 or z_1 has to nonzero, let $z_0 \neq 0$. We parametrize $\mathbb{CP}^1$ by $(z_0, z_1) \mapsto (z_1/z_0)$ and $(z_1/z_0) \mapsto (1, z_1)$. If $z_1 = re^{i\theta}$, we can turn f (when $m = 2$) into a real valued function on $\mathbb{R}^2$:

$$\begin{aligned}
f(r,\theta) &= \frac{|1^2 + r^2 e^{2i\theta}|}{(1^2+r^2)^m} \\
&= \frac{(1 + r^2\cos(2\theta))^2 + (r^2\sin(2\theta))^2}{(1+r^2)^2} \\
&= \frac{1 + 2r^2\cos(2\theta) + r^2}{(1+r^2)^2}.
\end{aligned}$$

This means our derivatives are

$$\frac{\partial f}{\partial \theta} = \frac{-4r^2\sin(2\theta)}{(1+r^2)^2},$$
$$\frac{\partial f}{\partial r} = \frac{2r(-2r^2\cos 2\theta - r^2 + 2\cos 2\theta - 1)}{(1+r^2)^3}.$$

This means that $(1,0)$ and $(1,1)$ are both critical points. Now are second order derivatives are

$$\frac{\partial^2 f}{\partial \theta^2} = \frac{-8r\cos(2\theta)}{(1+r^2)^2},$$

$$\frac{\partial^2 f}{(\partial r)(\partial \theta)} = \frac{\partial^2 f}{(\partial \theta)(\partial r)} = -\frac{8r\sin(2\theta)\left(-r^2+1\right)}{(1+r^2)^3},$$

$$\frac{\partial^2 f}{\partial r^2} = \frac{2\left(6r^4\cos(2\theta)+3r^4-16r^2\cos(2\theta)+2r^2+2\cos(2\theta)-1\right)}{(1+r^2)^4}.$$

Now if $r \neq 0$ satisfies $9r^4 - 14r^2 + 1 = 0$ (which does have a nonzero real solution) and $\theta = 0$, then the Hessian as a row of zeros making it singular. Therefore $(1,1)$ is degenerate so f is not a Morse function.

(8) We know that lines in $\mathbb{R}^2$ between lattice points become closed curves in $\mathbb{R}^2/\mathbb{Z}^2$. Therefore, the topology of the sublevel set are loops on the torus because of the sinusoidal functions.

(9) We can create two minimum points at opposite sides of the sphere and these have index 0. We can do the same for the maximum points and two different antipotal points. These points will have index 2. Finally, we create saddle points for the index 1 points.

(10) We claim that we can make a function such that $f^{-1}(y)$ has two elements for all $y \in \mathbb{R}$. This is essentially a question of mapping $\mathbb{R}$ to $2\mathbb{R}$ which can clearly be done since the cardinality of the both these sets are infinity of the same magnitude. For example we can map $-x$ and x to x for all positive x. This means that when y is positive, we see that $f^{-1}(y)$ has two elements and when y is negative, we see that $f^{-1}(y)$ has 0 elements.

(11) A Klein Bottle is a surface so it is a 2-manifold. Therefore, we can find a parameterization of the surface to get a function $f : \mathbb{R}^2 \to \mathbb{R}$ which we can clearly make a Morse function. For example, we can set $f(x) = \cos(2\pi x) + \cos(2\pi y)$.

(12) First we divide the noncompact manifold into an countably infinite number of compact pieces. Then we can find a function on X that only has isolated critical points (which always exists). Now we push each critical point from one compact piece to the next and keep doing this. At the limit, we will have a function with no critical points.

9 Week 9

(1) The only homology group that is defined is the 2nd homology group which is $H_2 = \ker(f_2)/\operatorname{im}(f_3)$ where $f_2 : A \to 0$ and $f_3 : 0 \to A$. This means that $\ker(f_2)/\operatorname{im}(f_3) = A/\{0_A\} \cong A$ where 0_A is the identity of A.

(2)

(a) Since the sequence is exact, the homotopy groups are trivial so $H_3 = \ker(f_3)/\operatorname{im}(f_4) = 0$ and $H_2 = \ker(f_2)/\operatorname{im}(f_3) = 0$. Therefore $\ker(f_3) = \operatorname{im}(f_4)$ and $\ker(f_2) = \operatorname{im}(f_3)$. Since $k_2 : B \to 0$, we see that $B = \ker(f_2) = \operatorname{im}(f_3)$ so f_3 is surjective. Next f_4 is homomorphism so $\operatorname{im}(f_4)$ is just the identity of A. Therefore, we have $\ker(f_3) = \operatorname{im}(f_4) = 0_A$, or the identity of A. This means that $\ker(f_3)$ is trivial so f_3 is injective. We have shown that f_3 is a bijection which is an isomorphism so $A \cong B$.

(b) If the sequence is exact, then $\ker(f_n) = \operatorname{im}(f_{n+1})$. Since $A \oplus C \cong B$, there exists isomorphisms $\tilde{f} : A \oplus C \to B$ and $\tilde{f}^{-1} : B \to A \oplus C$. Now let $A' = \{(a, 0) : a \in A\}$ so $A' \subseteq A \oplus C$. We can defined the maps $f : A \to B$ and $g : B \to C$ as f is the first entry of $\tilde{f}|_{A'}$ and g is the second entry of $\tilde{f}^{-1}$. Notice that the image of f is the kernel of g. Finally, we see that f is injective and g is surjective since $\tilde{f}$ and $\tilde{f}^{-1}$ are isomorphisms so the sequence is exact.

(c) In the previous problem, we used isomorphisms to prove something about homomorphisms. The fact that $A \oplus C \cong B$ gives us an isomorphism and we used it to create homomorphisms for our exact sequences. Therefore it is natural that the other way doesn't work. Our definitions of f and g don't imply that $\tilde{f}$ is an isomorphism since f was defined as a restriction of $\tilde{f}$. It could be the case that $\tilde{f}$ is an isomorphism on A' but nowhere else in A. This means that the sequence being exact is weaker than the condition $B \cong A \oplus C$.

(6) Note that every surface of genus g can be written as

$$S_g = \mathbb{T} \# \mathbb{T} \cdots \# \mathbb{T}$$

or the connected sum of g tori. Now $H_1(\mathbb{S}^1) = \mathbb{F}^2$ so by the Künneth formula, we have $H_1(\mathbb{T}) = H_1(\mathbb{S}^1 \times \mathbb{S}^1) = \mathbb{F}^4$. Finally, the homology of a connected sum is just the direct sum of the homologies so $H_1(S_g) = \mathbb{F}^{4g}$.