An Introduction to De Rham Cohomology Groups

Nandana Madhukara

Euler Circle

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Exterior Algebra

Tensors

Definition (Tensor)

A p-tensor on a vector space V is any real-valued function T such that on V^p that is multilinear i.e.

$$T(v_1, ..., v_j + av_j', ..., v_p) = T(v_i, ..., v_j, ..., v_p) + aT(v_i, ..., v_j', ..., v_p).$$

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Example

- A 1-tensor is a linear form so $\mathcal{J}^p(V^*) = V^*$. 1-tensor could be a measurement of the length of a vector.

Tensor Examples (contd.)

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Definition (Tensor Product)

If T is a p-tensor and S is a q-tensor, then $T \otimes S$ is a $p+q$ tensor:

$$T \otimes S(v_1, ..., v_p, u_1, ..., u_q) = T(v_1, ..., v_p) \cdot S(u_1, ..., u_q)$$

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Remark

Tensor product is not commutative!

Alternating Tensors

Definition (Alternating Tensor)

A p-tensor T is called *alternating* if $T = (-1)^{\pi} T^{\pi}$ where

$$T^{\pi}(v_1, v_2, \ldots, v_p) = T(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(p)})$$

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Proposition

$\mathrm{Alt}(T)$ *is indeed an alternating tensor for all p-tensors T.*

Alternating Tensors Examples

Example

- If T is a 1-tensor, every $\pi \in S_1$ is even so $T^\pi = T \implies T = (-1)^\pi T^\pi$.

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Remark

If $\Lambda^p(V^*)$ is set of alternating p-tensors, it's subspace of $\mathcal{J}^p(V^*)$.

Wedge Product

Definition (Wedge Product)

If $T \in \Lambda^p(V^*)$ and $S \in \Lambda^q(V^*)$, the *wedge product* is the $p+q$ tensor $T \wedge S \in \Lambda^{p+q}(V^*)$ defined by

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Proposition

If $\{\phi_1, ..., \phi_k\}$ is a basis of V^*, then $\{\phi_I : 1 \leq i_1 < \cdots < i_p \leq k\}$ is a basis of $\Lambda^p(V^*)$ where $I = (i_1, ..., i_p)$ and $\phi_I = \phi_{i_1} \wedge \cdots \wedge \phi_{i_p}$.

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Remark

The wedge prodcut is anticommutative i.e.

$$T \wedge S = (-1)^{pq} S \wedge T$$

Exterior Calculus

Differential Forms

Definition (Differential p-forms)

Let X be a smooth manifold. A differential p-form on X is a function ω that assigns each point $x \in X$ to an alternating p-tensor ω_x on the tangent space of X at x. This means that $\omega_x \in \Lambda^p(T_x(X)^*)$. The set of all p-forms on X is denoted by $\Omega^p(X)$.

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Remark

- Recall tensors = measuring devices for vector space
- In a differential form, we assign each point on a smooth manifold to these measuring devices
- Differential form = instructions for how to measure tangent vectors at each point on a manifold

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We can do operations on p-forms:

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- If $\phi : X \to \mathbb{R}$ is smooth, then $d\phi_x$ is 1-tensor on $T_x(X)$.

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Example

- If $\phi : X \to \mathbb{R}$ is smooth, then $d\phi_x$ is 1-tensor on $T_x(X)$.
- $x \mapsto d\phi_x$ defines the 1-form, $d\phi$, on X called *differential* of ϕ.

The Exterior Derivative

Proposition

Let $x_1, \dots, x_k$ be coordinate functions for $\mathbb{R}^k$ and $dx_I = dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ where $I = (i_1, \dots, i_p)$. Every p-form on an open set $U \subseteq \mathbb{R}^k$ can be uniquely written as $\sum_I a_I dx_I$ where the sum ranges over all increasing index squences I and the a_I are 0-forms on U.

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Definition (Exterior Derivative)

Let $\omega = \sum_I a_I dx_I$ be a smooth p-form on an open set of $\mathbb{R}^k$. The exterior derivative of ω is the $(p+1)$-form

$$d\omega = \sum_I da_I \wedge dx_I.$$

The Exterior Derivative Rules

Remark

Can define exterior derivatives on a smooth manifold X by considering coordinate charts.

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- Product Rule (where $\omega \in \Omega^p(X)$):
$$d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^p \omega \wedge d\theta$$
- Cocycle condition:
$$d(d\omega) = 0$$

The Pullback Map

Definition (Pullback Map)

If $f : X \to Y$ be a smooth map and let $df_x : T_x(X) \to T_{f(x)}(Y)$ be the derivative. The linear map $f^*\omega : \Omega^p(Y) \to \Omega^p(X)$ is called the *pullback by f at x* and maps a p-form on Y, ω, to a p-form on x, $f^*\omega$, defined by

$$(f^*\omega)_x(v_1, ..., v_k) = \omega_{f(x)}(df_x(v_1), ..., df_x(v_k)).$$

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Proposition

The pullback commutes with the exterior derivative.

De Rham Cohomology Groups

Closed and Exact Forms

Definition

A p-form ω on X is *closed* if $d\omega = 0$ and *exact* if $\omega = d\theta$ for some $(p-1)$-form θ. The set of all closed p-forms on X is denoted by $Z^p(X)$ and the set of all exact p-forms on X is denoted by $B^p(X)$

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Definition

Two closed p-forms ω, ω' are called *cohomologous*, denoted by $\omega \sim \omega'$ if $\omega - \omega'$ is exact.

De Rham Cohomology Groups

Definition (De Rham Cohomology Groups)

Consider the following sequence

$$0 \to \Omega^0 \xrightarrow{d^0} \Omega^1 \xrightarrow{d^1} \Omega^2 \xrightarrow{d^2} \cdots$$

which we call *cochain complex* where d^p is the exterior derivative on p-forms. The pth *De Rham cohomology group* (or pth *cohomology group* for short) is $H^p(X) = \ker(d^p) \setminus \operatorname{im}(d^{p-1})$.

An element of $H^p(X)$ is called a *cohomology class* and the cohomology class containing the p-form ω is denoted by $[\omega]$ i.e.

$$[\omega] = \{\omega + d^{p-1}\omega' : \omega' \in \Omega^{p-1}\}.$$

Homotopy Invarience

The Theorem

Theorem (Homotopy Invarience)

If X and Y are homotopy equivalent smooth manifolds, then their pth cohomology groups are isomorphic for every p i.e. $H^p(X) \cong H^p(Y)$.

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Definition

If $f : X \to Y$ is a smooth map, the pullback f^* creates the *induced cohomology map* (still denoted by f^*) from $H^p(Y)$ to $H^p(X)$:

$$f^*[\omega] = [f^*\omega].$$

The Proof

Remark

Homotopic smooth maps induce the same cohomology map.

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If $f, g : X \to Y$, what does it mean for $f^* = g^*$?

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If $f, g : X \to Y$, what does it mean for $f^* = g^*$?

Answer

- If $f^* = g^*$, then

$$f^*\omega - g^*\omega = d\theta$$

since this would mean
$f^*[\omega] - g^*[\omega] = [f^*\omega] - [g^*\omega] = [d\theta] = 0$ or the identity.

The Proof (contd.)

Answer (contd.)

- We can generate the θ with $h : \Omega^p(Y) \to \Omega^{p-1}(X)$ so condition becomes

$$f^*\omega - g^*\omega = d(h\omega) + h(d\omega).$$

The Proof (contd.)

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If X is a smooth manifold and $t \in I$, let $i_t : X \to X \times I$ be the map $i_t(x) = (x, t)$.

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Definition

If X is a smooth manifold and $t \in I$, let $i_t : X \to X \times I$ be the map $i_t(x) = (x, t)$.

Lemma

For any smooth manifold X, there exists a homotopy operator between $i_0^*, i_1^* : \Omega^p(X \times I) \to \Omega^p(X)$ for every p.

The Proof (contd.)

Lemma

If X and Y are smooth manifolds and $f, g : X \to Y$ are homotopic smooth maps. For every p, the induced cohomology maps $f^*, g^* : H^p(Y) \to H^p(X)$ are equal.

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Proof.

- By previous lemma i_0^* and i_1^* are equal.

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- Since f and g are homotopic smooth maps, they are smoothly homotopic so let $H : X \times I \to Y$ be the smooth homotopy.

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- Since f and g are homotopic smooth maps, they are smoothly homotopic so let $H : X \times I \to Y$ be the smooth homotopy.
- $f = H \circ i_0$ and $g = H \circ i_1$
- $$f^* = (H \circ i_0)^* = i_0^* \circ H^* = i_1^* \circ H^* = (H \circ i_1)^* = g^*.$$

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The Proof (contd.)

Proof of Homotopy Invarience.

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$$(\tilde{f} \circ \tilde{g})^* = (\mathrm{Id}_Y)^* \implies \tilde{g}^* \circ \tilde{f}^* = \mathrm{Id}_{H^p(Y)}$$

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- Similarly, we have $\tilde{g} \circ \tilde{f} \simeq g \circ f \simeq \mathrm{Id}_X$ so $\tilde{f}^* \circ \tilde{g}^* = \mathrm{Id}_{H^p(X)}$.

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- By second lemma,

$$(\tilde{f} \circ \tilde{g})^* = (\mathrm{Id}_Y)^* \implies \tilde{g}^* \circ \tilde{f}^* = \mathrm{Id}_{H^p(Y)}$$

- Similarly, we have $\tilde{g} \circ \tilde{f} \simeq g \circ f \simeq \mathrm{Id}_X$ so $\tilde{f}^* \circ \tilde{g}^* = \mathrm{Id}_{H^p(X)}$.
- Therefore, the map $\tilde{f}^* : H^p(Y) \to H^p(X)$ is an isomorphism.

■

Applications

Corollary (Topological Invarience)

If X and Y are homeomorphic smooth manifolds, then their cohomology groups are isomorphic i.e. the cohomology groups are topologically invarient.

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The whole point of De Rham cohomology groups was to investigate when closed forms are exact.

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If X and Y are homeomorphic smooth manifolds, then their cohomology groups are isomorphic i.e. the cohomology groups are topologically invarient.

Recall

The whole point of De Rham cohomology groups was to investigate when closed forms are exact.

Definition

A star-shaped set U is a set where there exists a $c \in U$ such that for every $x \in U$, the line segment between c and x is contained in U.

Poincaré Lemma

Theorem (Poincaré Lemma)

If U is a star-shaped open subset of $\mathbb{R}^n$, *then* $H^p(X) = 0$, *or the trivial group, for* $p \geq 1$.

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Proof.

- Every point on X has neighborhood diffeomorphic to open ball in $\mathbb{R}^n$
- $\mathbb{R}^n$ is star-shaped and cohomology groups are diffeomorphically invarient so we're done by Poincaré Lemma.

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Lemma

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- $c_x \circ \iota_x = \mathrm{Id}_{\{x\}}$ and $\iota_x \circ c_x \simeq \mathrm{Id}_X$ so ι_x is homotopy equivalence
- By homotopy invarience, $H^p(X) = H^p(\{x\}) = 0$ since $\{x\}$ is a 0-manifold.

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- A star-shaped domain is contractible because of straight-line homotopy:

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- A star-shaped domain is contractible because of straight-line homotopy:
$$H(x,t) = c + t(x-c).$$
- By previous lemma, we have $H^p(U) = 0$.

∎

The End

Fin.