More Counting (with combinations)!

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Combinations and Applications

A Question about Pizza

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- **Total**: $6 \times 5 \times 4 = \boxed{120}$

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$$\frac{6 \times 5 \times 4}{3 \times 2 \times 1} = \frac{120}{6} = \boxed{20}$$

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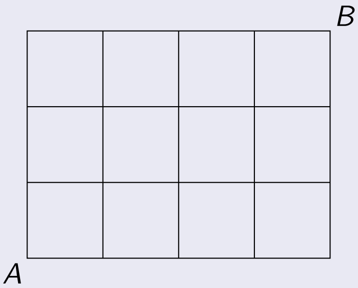
- **Total**:

$$\frac{n \times (n-1) \times \cdots \times n-(k-1)}{k \times k-1 \times \cdots \times 1} = \frac{n!}{k!(n-k)!} = \boxed{\binom{n}{k}}$$

Path Counting

Question

How many ways are there to go from A to B in the grid below by only going up and to the right.



Practice Problems

Problem

The Senate has 100 members, consisting of 55 Republicans and 45 Democrats. In how many ways can I choose a 5-person committe consisting of 3 Republicans and 2 Democrats?

Problem

Consider a regular octagon. How many triangles can be formed whose vertices are the vertices of the octagon?

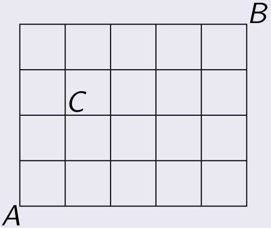
Problem

How many triangles can be formed whose vertices are points in a 5×5 *square grid of points?*

Practice Problems Contd.

Problem

How many points are there from A to B passing through C?



Problem

Nine lines are drawn in a plane. What is the largest possible number of points in the plane at which at least two of the nine lines intersect?

More Challenging Problems

Problem

The sundae bar at Sarah's favorite restaurant has 5 toppings: hot fudge, sprinkles, walnuts, cherries, and whipped cream. In how many different ways can Sarah top her sundae if she is restricted to at most 2 toppings?

Problem

There are 5 different pairs of gloves, where left and right are distinguishable. Select 4 from the 10 gloves.

- *How many ways are there to select 2 pairs of gloves?*
- *How many ways are there to select 4 such that some 2 of the 4 make a pair?*

Challenging Problems contd.

Problem

In poker, a 5-card hand is called a three of a kind if there are three cards of one rank and two other cards which are not the same rank as each other or as the other three cards. How many 5-card hands are three of a kind?

Problem

Use combinations to find the number of distinct arrangements of the letters of 'ONONONONO'

Challenging Problems contd.

Problem

Let ABCDEFGH be a cube

- *How many different line segments can be formed by connecting the vertices of the cube?*
- *How many different triangles can be formed by connecting 3 of the vertices of the cube?*
- *How many noncongruent triangles can be formed by connecting 3 of the vertices of the cube?*

Problem (AIME)

An integer is called snakelike if its decimal representation $a_1a_2a_3\cdots a_k$ *satisfies* $a_i < a_{i+1}$ *if* i *is odd and* $a_i > a_{i+1}$ *if* i *is even. How many snakelike integers between 1000 and 9999 have four distinct digits?*

Pascal's Triangle

Some Interesting Things

Remark

The rows of Pascal's Triangle are combinations.

Identity (Pascal)

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}.$$

Proposition

$$\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Practice Problems

Problem

Prove that

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

using a committe-forming argument.

Problem

Prove the identity

$$\binom{n-1}{k-1} = \frac{k}{n}\binom{n}{k}$$

- *by algebra.*
- *by a committe-forming argument.*

Challenging Problems

Problem

Find a formula for

$$\binom{n}{0}\binom{n}{1}+\binom{n}{1}\binom{n}{2}+\binom{n}{2}\binom{n}{3}+\cdots\binom{n}{n-1}\binom{n}{n}$$

Problem (AIME)

Find the smallest value of n such that Row n of Pascal's Triangle contains three successive entries with the ration 3:4:5.

Problem

Try to find a way to generate the Fibonacci numbers from Pascal's triangle. Why does your way work?

Binomial Theorem

The Theorem

Problem

Expand the following:

- $(x+y)^2$
- $(x+y)^3$
- $(x+y)^4$
- $(x+y)^5$
- *Do you see a pattern?*

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Theorem (Binomial Theorem)

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \cdots + \binom{n}{n}y^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k.$$

Practice Problems

Problem

Prove

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Problem

Find a formula for

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}$$

Problem

Find a formula for

$$\sum_{k=0}^{n} 2^k \binom{n}{k}$$

Practice Problems

Problem

What is the coefficient of the x^{11} *term in*

$$\left(\frac{x^2}{2} - 3x\right)^7?$$

Problem

Write $(3 - 2\sqrt{5})^5$ *in the form* $a + b\sqrt{5}$ *for some integers* a *and* b.

Problem

Compute

$$\binom{16}{0} + \binom{16}{2} + \binom{16}{4} + \cdots + \binom{16}{16}.$$

Challenging Problems

Problem

Can you find the general expression for the coefficient of the $x^i y^j z^k$ *term in the trinomial* $(x+y+z)^n$*?*

Problem

Compute the sum

$$\binom{20}{20} + \binom{20}{18}\left(\frac{1}{2}\right)^2 + \binom{20}{16}\left(\frac{1}{2}\right)^4 + \cdots + \binom{20}{0}\left(\frac{1}{2}\right)^{20}$$

Problem

How many terms are in the expansions of:

- $(a+b+c)^8$
- $(a+b+c)^8 + (a+b-c)^8$

The End

Fin.