Problem Set Solutions for

# Combinatorics

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## Contents

# 1 Week 1

**(1)** We know that each there must be only one rook in each row and column because the rooks would attack each other if this was not the case. If we go row by row, there are $n$ squares to place the first rook, $n-1$ square to place the second rook (we can't place it in the same column as the first rook, and so on. Therefore there are $n \cdot (n-1) \cdot (n-2) \cdots 1 = \boxed{n!}$ ways of placing $n$ rooks.

**(2)** We can use $k-1$ bars to split up $n$ stars into $k$ groups. The number of stars in the $i$th region corresponds to $x_i$. Therefore we have

$$\binom{n+k-1}{k} = \boxed{\left(\!\binom{n}{k}\!\right)}$$

solutions. Since the number

**(3)** To have a successful arrangement, for the $i$th person with a 10 dollar bill, there must be at least $i$ people with 5 dollar bills in front of the person. We do this by first paring up the the 5 dollar and 10 dollar people in $n!$ ways. Then we arrange all $2n$ people such that for each pair, the five dollar person is always in front of the 10 dollar person.

Now we count the number of ways of doing this. For the $k$th pair, there will be $2n-2(k-1) = 2n-2k+2$ places to choose the two spaces for the pair. Therefore there are

$$\binom{2n-2k+2}{2}$$

ways of placing the pair since there is only one way they can be arranged such that the 5 dollar person is in front of the 10 dollar person. Since picking where to place each pair is independent of each other, we take the product over all possible values of $k$ to get

$$\prod_{k=0}^{n} \binom{2n-2k+2}{2}$$

ways of arranging the pairs. Now we use the fact that there are $n!$ ways of picking the pairs to get our answer:

$$\boxed{n! \prod_{k=0}^{n} \binom{2n-2k+2}{2}}$$

**(4)** We know that

$$\binom{2n}{n} = \frac{(2n)!}{(n!)(n!)}.$$

Therefore, if we prove that $p$ divides $(2n)!$ the same number of times as $(n!)^2$, we are done. Since $p$ is prime and $p < n$, we know that $p$ divides $n!$ once so $p$ divides $(n!)^2$ twice. Similarly we can see that $p$ divides $(2n)!$ twice since $2n/3 < p \le n$ so we are done.

**(5)** Let $a = 1 + 1 + \cdots + 1$. By the Multinomial Theorem

$$a^p = (1+1+\cdots+1)^p = \sum_{k_1+\cdots+k_a=n} \binom{p}{k_1, \ldots, k_d} = \sum_{k_1+\cdots+k_a=n} \frac{p!}{k_1!k_2!\cdots k_a!}.$$

Since $p$ is a prime, the only times when the multinomial does not divide $p$ is when $k_i = p$ for $i = 1, 2, ..., a$. This happens $a$ times so $a^p \equiv a \bmod p$.

**(6)** There are $\binom{n}{k}$ ways of picking a $k$ element subset out of $[n]$. We let this be $A$. Then we pick a non-empty subset out of $A$ to be the intersection set of $A$ and $B$ in $2^k - 1$ ways. Finally we pick the rest of the elements of $B$ in $2^{n-k}$ ways. Summing over all values of $k$ we get

$$\sum_{k=0}^{n} \binom{n}{k} (2^k - 1) 2^{n-k} = \boxed{\sum_{k=0}^{n} \binom{n}{k} (2^n - 2^{n-k})}.$$

**(11)** We use complementary counting and count the number of partitions with at least one part being 1. We construct our partitions by first having a 1 and then there will be $p(n-1)$ partitions. Therefore we have a total of $\boxed{p(n) - p(n-1)}$ partitions without a 1.

**(14)** When we connect all numbers that are not multiples of $p$ in Pascal's Triangle, we see that they form a Sierpinski Triangle. So all multiples of $p$ lie in the spaces of the Sierpinski Triangle.

**(16)** By the binomial theorem

$$\begin{aligned}
\sum_{k=0}^{m+n} \binom{m+n}{k} x^k &= (1+x)^{m+n} \\
&= (1+x)^m (1+x)^n \\
&= \left( \sum_{i=0}^{m} \binom{m}{i} x^i \right) \left( \sum_{k=0}^{m} \binom{m}{k} x^k \right) \\
&= \sum_{r=0}^{m+n} \left( \sum_{k=0}^{r} \binom{m}{i} \binom{n}{r-i} \right) x^r
\end{aligned}$$

By comparing coefficients we get

$$\sum_{k=0}^{r} \binom{m}{i} \binom{n}{r-i} = \binom{m+n}{k}$$

and we are done.

**(17)** Let $n_i$ for $i = 1, 2, .., d$ be the number of animals of some species. (The species are all distinct). The RHS counts the number of ways of picking $r$ animals out of all the animals. The LHS counts the same thing by picking $k_i$ elements from each species such that $k_1 + k_2 + \cdots + k_d = r$. Therefore the LHS and RHS are the same.

# 2 Week 2

**(1)** Since $N$ is square-free, all the powers of the prime factors are 1. Therefore our question is how to split the set of prime factors of $N$ into set partitions. The answer to this for $n$ prime factors is the $n$th Bell number $\boxed{B_n}$.

**(2)** We first start off by partitioning $[n]$ into 3 set partitions. We can do this in

$$\left\{ n \atop 3 \right\}$$

ways. Now we arrange the set partitions such that there are exactly 2 descents. First we arrange the set partitions in any way. We can do this in $3!\left\{ n \atop 3 \right\}$ ways and since we have a order now, let the first set be $A$, the second set be $B$, and the third set be $C$. Now we subtract the number of ways to arrange the set partitions such that there are 0 accents and the number for 1 accent.

The former can counted by counting the number of ways to create $A$, $B$, and $C$ such that for some $j$ and $k$,

$$\begin{aligned} A &\equiv \{x \mid x \leq j\} \\ B &\equiv \{x \mid j < x \leq k\} \\ C &\equiv \{x \mid x > k\} \end{aligned}$$

We do this by listing out $[n]$ in numerical order and then placing 2 bars in between the numbers. The groups of numbers created by the bars are our set partitions. Since each set partition must be non empty $n-1$ places to place the bars so we have a total of

$$\binom{n-1}{2}$$

ways of having 0 descents.

The latter can be counted in

$$\left\langle {n \atop 1} \right\rangle = 2^n - n + 1$$

ways by definition. Putting all this together gives us a total of

$$\boxed{3!\left\{ {n \atop 3} \right\} - \binom{n-1}{2} - 2^n + n + 1}.$$

**(3)** We start by proving $\left\{ {2n \atop n} \right\} > \binom{2n}{n}$. The LHS is the number of ways of creating $n$ set partitions out of $[2n]$. The RHS is the number of ways of picking the largest element in each of the set partitions. Therefore it makes sense that the LHS is greater than the RHS because after we pick the largest element there are more ways of arranging the other elements of $[2n]$, so

$$\left\{ {2n \atop n} \right\} > \binom{2n}{n} > n!$$

Finally, we can see that $\left\{ {2n \atop n} \right\} < (2n)!$ since we have more freedom when permuting $[2n]$ than when creating set partitions. Therefore we are done.

**(4)** There are $B_n$ total set partitions and there must be $B_{n-1}$ set partitions with at least one block of size one. Therefore our answer is $\boxed{B_n - B_{n-1}}$.

**(5)** If $n$ is odd, then the two parts are always different in size so our answer is $\boxed{\left\{ {n \atop 2} \right\}}$. If $n$ is even, we use complementary counting. The total number of set partitions with two parts is $\left\{ {n \atop 2} \right\}$. Then the number of set partitions with parts of the same size are $\binom{n}{n/2}$ since we pick half of them to be in one part and the other half in the other part. Additionally, we must divide this binomial coefficient by two since we do not care about order. Therefore we have a total of

$$\boxed{\left\{ {n \atop 2} \right\} - \frac{1}{2}\binom{n}{n/2}}.$$

**(6)** When 1 and 2 are in the same cycle, each other element of $[n]$ can either be in the $1-2$ cycle or out of it. If there are $k$ elements not in the $1-2$ cycle, then there are $k!$ ways of arranging these outer elements. Then there are $n-k$ elements in the $1-2$ cycle so there are $(n-k-1)!$ ways of arranging the $1-2$ cycle. Summing over all possible values of $k$ gives us

$$\boxed{\sum_{k=0}^{n-2} k!(n-k-1)!}.$$

This same thing happens for our $1,2,3$ in the same cycle case except our upper bound for $k$ is $n-3$:

$$\boxed{\sum_{k=0}^{n-3} k!(n-k-1)!}.$$

**(8)** We start by noticing that the $a_n$ in our formula corresponds to the set partitions where there are no blocks of size 1. Therefore we must now prove that $a_{n+1}$ is the number of set partitions of $[n]$ with blocks of size 1. The way we make this bijection is by first looking at any set partition of $[n]$ with at least one block of size 1. Then we add $n+1$ to a set partition of size 1. If there are no more blocks of size 1, we are done. If not, we merge all the blocks of size 1. This will uniquely convert a set partition of $[n]$ with at least one block of size 1 into a set partition of $n+1$ with no blocks of size 1, so we have a bijection.

**(11)** Since $f$ and $g$ each produce $n$ cycles of size 2, when we combine these two permutations for the cycles of any length always come in pairs. Therefore the number of cycles of length $k$ in $f(g([n]))$ is even.

**(14)** The LHS by definition is the number of ways of partitioning $[n]$ into $k$ set partitions. For the RHS, we first look at the sum. Let us start from the end of the sum and work ourselves back to the beginning. The last terms tells that out of $k$ boxes, we will pick all of them and we will split the elements of $[n]$ among the boxes. This is however an over count since are counting the the case when $k-1$ of the boxes have elements but one does not. Therefore we subtract of this case, so we subtract

$$\binom{k}{k-1}(k-1)^n$$

which is our second to last term. However this is an under count since we are subtracting the case when $k-2$ boxes have elements but two do not too many times. Therefore we add this back:

$$\binom{k}{k-2}(k-2)^n.$$

And we keep going like this: alternating between over counting and under counting until we hit $i=0$. Finally, we have been counting ordered set partitions so we divide the whole sum by $k!$ factorial and now we have our full RHS. Therefore the LHS and RHS are equal.

**(17)** The card trick requires for an assistant. This person knows a code which is a bijective function that maps set partitions of [5] to cards in a deck. This works since $B_n = 52$ and there are 52 cards in a deck. Then the assistant picks a card. After this is done, they send a text to you, who is sitting in a room, and say "I have gotten the card". They partition these 5 sets of words using the code where two spaces is when a new set of words begins. Using the same code, you decipher the message and guess the card correctly.

**(21)** Using a computer program we get the answer for $n$. Here is the python program:

```
import itertools

def avoidence(n):
    """Returns the number of permutations of n avoiding 123 and 231"""
    counter = 0
    n_list = [_ for _ in range(1, n+1)]
    perm_list = list(itertools.permutations(n_list))
    for perm in perm_list:
        flag = True
        for a in range(n):
            for b in range(a+1, n):
                for c in range(b+1, n):
                    if ((perm[a] < perm[b] and perm[b] < perm[c])
                    or (perm[a] < perm[b] and perm[b] > perm[c])):
                        flag = False
            break
        if flag:
            counter += 1
    return counter
```

**Answer**: $n! - (n-1)!$

# 3 Week 3

**(1)** Let $A_i$ be the set of functions such that $f(i) = i$. If $F$ is the set of all of all functions $f : [n] \to [n]$, we want to count

$$|F| - \left|\bigcup_{i=1}^{n} A_i\right|.$$

We first start with $|F|$. The $n$ integers in the domain each have $n$ choices so we have a total of $n^n$. Now let us move on to the union. We calculate this using the inclusion-exclusion principle. Since each $A_i$ is the same size, the way we find each part of the inclusion and exclusion is by finding the part of $A_n$ and then generalizing for $A_i$.

Let us start by finding $|A_n|$. We know that $f(n) = n$ so the rest of $f$ is another function $g : [n-1] \to [n]$ so $|A_n| = n^{n-1}$. Similarly $|A_i| = i^{i-1}$.

Next is $|A_{n-1} \cap A_n|$. This means that $f(n) = n$ and $f(n-1) = n-1$, so the rest of $f$, similar to last time, is $g : [n-2] \to [n]$. Therefore $|A_{n-1} \cap A_n| = n^{n-2}$ and in general $|A_{i-1} \cap A_i| = i^{i-2}$. IF we keep going like this and plug everything in, we get

$$\begin{aligned}\left|\bigcup_{i=1}^{n} A_i\right| &= n \cdot n^{n-1} - \binom{n}{2} n^{n-2} + \cdots + (-1)^{n+1} \binom{n}{n} n^0 \\ &= \sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} n^{n-i}\end{aligned}$$

By the Binomial Theorem, we know that

$$\sum_{i=0}^{n} \binom{n}{i} (-1)^i n^{n-i} = (n-1)^n.$$

Now we can negate both sides and add $n^n$ to get

$$\sum_{i=1}^{n} (-1)^{i+1} \binom{n}{i} n^{n-i} = n^n - (n-1)^n.$$

This means our full answer is

$$n^n - \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i} n^{n-i} = \boxed{(n-1)^n}.$$

A much easier way of counting this is saying that each of the $n$ integers in the domain of $f$ cannot go to $n$ in the range so they each have $n-1$ choices. Therefore our total is $\boxed{(n-1)^n}$ functions which is sure enough the answer above.

**(2)** There are $4^4$ ways of picking a specific jack, queen, king, and ace. Now that we have at least one jack, queen, king, and ace in our hand, we can fill up the 9 remaining spots in our hand with any of the 48 cards that are left in the deck. We just choose 9 cards out of remaining deck and in $\binom{48}{9}$ so our answer is

$$\boxed{4^4 \binom{48}{9}}.$$

**(3)** Let $A_i$ be the set of integers from 1 to 10000 that are divisible by $i$. First of all, note that

$$|A_i| = \left\lfloor \frac{10000}{i} \right\rfloor$$

By the Principle of Inclusion and Exclusion, we must find

$$\begin{aligned}
&|A_4 \cap A_5| + \cdots + |A_6 \cap A_7| \\
&- 3(|A_4 \cap A_5 \cap A_6)| \cdots |A_5 \cap A_6 \cap A_7|) \\
&+ 6|A_4 \cap A_5 \cap A_6 \cap A_7|
\end{aligned}$$

Using the lcm function to simplify the intersections and the formula we got above, we get the answer of $\boxed{1475}$.

**(7)** Let $f$ be a permutation of $[n]$ which is also an involution. By definition, the elements of $[n]$ can follow one of the two properties

1. $f(x) = k$ where $k \neq x$ and $f(k) = x$
2. $f(x) = x$

For the first property, let the set $\{x, k\}$ be called an "X" and for the second property, let the set $\{x\}$ be called a "bar." Therefore permutations that are involutions contain only X's and bars. Now we have two cases: $n$ is even and $n$ is odd.

If $n$ is even, then the number of bars must be even. Therefore the number of permutations with $k$ bars that are involutions (where $k$ is even) is $\binom{n}{k}$ since we choose $k$ elements out of $[n]$ to be bars. Therefore the total number of permutations is

$$\binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{n} - 1.$$

We subtract 1 since there is one trivial involution. This sum is $2^{n-1} - 1$ which is odd.

Now we $n$ is odd we get a similar sum:

$$\binom{n}{1} + \binom{n}{3} + \cdots + \binom{n}{n} - 1 = 2^{n-1} - 1$$

which is also odd. Therefore we are done

**(8)** Let $f(x) = k$ for some $x \in X$. Since $f$ is a bijective function, we know that there exists no other $x' \in X$ such that $f(x') = k$. Therefore the values of $f^n(x)$ will never go in a cycle, so $f^n(x)$ will eventually land on $x$.

This is however not true when the size of $X$ is infinite since even though $f^n(x)$ cannot go in a cycle $f^n(x)$ still has an infinite number of elements to go through. So there might be a function that always maps $x \in X$ to a new $x' \in X$ which would mean that $f^n(x) \neq x$ for any $n$. On the other hand, when $|X|$ is finite, $f^n(x)$ will always run out of elements and land on $x$.

**(9)** Plugging $b_k$ in gives us

$$\begin{aligned}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}a_j &= \sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^{n-k}\binom{n}{k}\binom{k}{j}a_j \\ &= \sum_{k=0}^{n}\sum_{j=0}^{k}(-1)^{n-k}\binom{n}{j}\binom{n-j}{k-j}a_j\end{aligned}$$

We cannot do anything with the binomials because of the $a_j$ so we switch the sums and factor out the $a_j$.

$$\sum_{j=0}^{n} a_j\binom{n}{j}\sum_{k=j}^{n}(-1)^{n-k}\binom{n-j}{k-j}$$

Now we change the index of the inner sum so that we can use the Binomial Theorem.

$$\sum_{j=0}^{n} a_j\binom{n}{j}\sum_{k=0}^{n-j}(-1)^{k}\binom{n-j}{k}$$

Now by the Binomial Theorem, the inner sum is just 0 for all $j \neq n$. When $j = n$, we get $0^0$ which is undefined. Therefore the RHS is just the sum evaluated at $j = n$:

$$\binom{n}{n}a_n = a_n.$$

Therfore we are done.

**(13)** We split this proof into two parts: converting partitions with distinct parts into partitions with odd parts and converting partitions with odd parts into partitions with distinct parts.

Distinct to Odd:

We first express each part uniquely as a power of 2 times an odd number. Then we factor out the odd numbers and we get a partitions with only odd parts.

Odd to Distinct:

We first group all the odd parts and express $n$ as $n = a_1 \cdot 1 + a_2 \cdot 3 \cdots$. Then write each $a_i$ as the sum of power of 2. Finally we distribute the odd numbers to get a partition of only distinct parts. This is because every integer can be uniquely written as a power of 2 times an odd number.

# 4 Week 4

(1) First we pick the rows in $\binom{n}{k}$ ways and the columns in $\binom{m}{k}$ ways. Finally we permute the rows and columns in $k!$ ways so our rook polynomial is

$$\boxed{\sum_{k=0}^{n} \binom{n}{k}\binom{m}{k} k! x^k}.$$

(2) For the first question, we do this using the method of Theorem 4.3. We first see that we need to append 4 zeros to our Ferrer's board so that it is $(0, 0, 0, 0, 1, 1, 1, 2, 4, 8)$. Now we calculate the $s_i$'s and we get the list $(0, -1, -2, -3, -3, -4, -5, -5, -4, -1)$ where the $i$th element of this list is $s_i$. Therefore we get $a_1 = 2$, $a_2 = 1$, $a_3 = 2$, $a_4 = 2$, $a_5 = 2$, and $a_6 = 0$. Finally by Theorem 4.3, the total number of Ferrer's boards is

$$\binom{2}{1}\binom{2}{2}\binom{3}{2}\binom{3}{2} = \boxed{18}.$$

Using the technique in the last paragraph of the chapter, we get that the Ferrer's board with distinct columns that has the same rook polynomial is $(1, 3, 5, 8)$.

(3) We see that $N_B(1)$ is just the sum of the coefficents so it must be the total number of permutations of $[n]$. Additionally, I could find with the OEIS that $r_B(1)$ for $B \subseteq [n] \times [n]$ is "the number of partial permutations of $[n]$." I also suspect that $N_B(-1)$ has to do with the Principle of Inclusion and Exclusion. With the examples I have tried I got $n - 1$.

(6) To show that the rook polynomials are the same, we show that the coefficients of $x^k$ in $r_A(x)$ and in $r_B(x)$ are the same. First let $r_i$ be the rook numbers for board $A$ and $R_j$ be the rook numbers for $B$. We first start with the rook polynomial $r_{A+[m]\times[n]}(x)$. We first put $i$ rooks in the unappended Ferrer's board in $r_i$ ways by definition. Then we put the $k - i$ rooks in the $[m] \times [n]$ board. We can do this in $n^{\underline{k-i}}$ since we can place the first rook in $n$ ways, the second rook in $n - 1$ ways and so on. Summing over all $i$ gives us

$$\sum_{i=0}^{k} r_i n^{\underline{k-i}}.$$

Similarly we get this for $r_{B+[m]\times[n]}(x)$:

$$\sum_{j=0}^{k} R_j n^{\underline{k-i}}.$$

Since $r_A(x)$ and $r_B(x)$ are the same we are done.

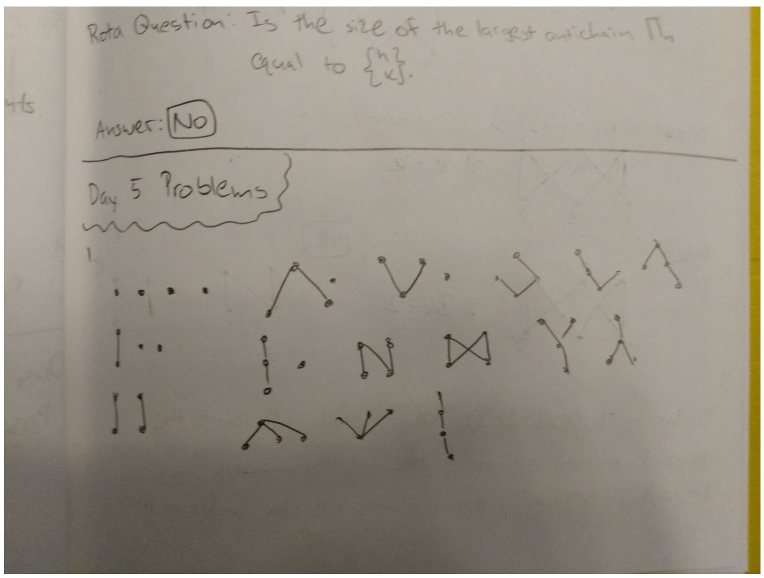
(7) We can see that $\boxed{N_B(1) = N_{B^c}(1)}$. Because of problem (3), the LHS counts the total number of permutations of $[n]$. Since $B^c \subseteq [n] \times [n]$, we know that $N_{B^c}(1)$ which is our RHS also counts the total number of permutations of $[n]$

(8) Since $r_{B^c}(x)$ is a hard function to deal with we can use the $N_{B^c}$ function. However I do not know how to go on from here. I suspect that I must relate this function to $N_B(x)$ but I do not know how to do so.

(12) A lot of the theory of rook polynomials can be brought over to three dimensions; however, instead of the rook being able to attack in two directions, it can attack in three directions. This will change our theorems but to fix this we can add extra factors to fix this. One thing that would be interesting is card matching. Instead of placing cards just on a table and matching them, we place them in cells of a box so that they can be matched 2 ways.

# 5 Week 5

**(1)**




**(2)** We do casework on where the 3 is. But before we do this, note that the 1 and 2 must be on the bottom and the 5 and 6 must be on the top. Our first case is 3 on the bottom. If this happens we can freely arrange the top and bottom independently each in 3! ways so our total is $3! \cdot 3! = 36$.

Now if the 3 is on the top, the 1 and 2 have to stay under the 3 and the 4 goes in the remaining spot in the bottom. So now there are 3 places the 3 can go and for each of these, the 5 and 6 on the top and the 1 and 2 in the bottom are arranged in 2 ways. Therefore we have a total of $3 \cdot 2 \cdot 2 = 12$ total ways. Adding this with our first case gives us an answer of $36 + 12 = \boxed{48}$.

**(3)** Similar to $N([2] \times [n])$, we can see that $N([3] \times [n])$ will count the three dimensional version of the Catalan Numbers. Instead of staying under a line, we stay under a plane. The reason we count the 3D Catalan Numbers since $[3] \times [n]$ has one more chain than the $[2] \times [n]$ which corresponds to a third dimension.

**(5)** Since $P$ is the set of *finite* permutations, for any permutation we will eventually run out of patterns to avoid. Therefore there cannot be an infinite antichain.

**(6)** The only way $f^{-1}$ cannot preserve order is if $f(P)$ introduces some new orders between the elements. That is if $f(P)$ makes new segments in the Hasse Diagram of $P$. We must prove that this never happens.

Since $F$ is a bijection from $P$ to itself, every time we add an edge, we must lose an edge so the total number of edges is constant. Therefore we are done. Note that this proof depends on the finiteness of $P$. If $P$ is infinite, a constant number of edges will not necessarily mean anything so we can add how many ever edges we want.

**(8)** We can have an infinite number of chains and antichains to create an infinite poset. For example, the poset with an infinite number of disjoint chains of length 2 is an infinite poset with finite chains and antichains.

**(9)** The chain where we start with a set of 1 number and we continue adding on numbers into this set is countable. I still do not understand how we can make an uncountable chain when we are dealing with Natural Numbers. The definition of countability itself comes from $\mathbb{N}$.

**(10)** What we want to do to checkmate the king is to trap it with the other black pieces. One thing I could figure out is that the king cannot end up in row 8 since the only way to block the king is with the rook since the pawns can only be in row 7 or lower. However, the rook will not be enough since it can only block one square. Therefore the king has to be below row 8.

**(15)** Let $R(n, m)$ be the maximum number of regions formed by $m$ hyperplanes in $\mathbb{R}^n$. Now we form a recurrence. Let us first start with $m-1$ hyperplanes, and the most number hyperplanes is $R(n, m-1)$. Now we add in a hyperplane, say $H$, into our existing set. Now the number of regions it adds must be equal to the number of regions the other hyperplanes split $H$ into. Since $H$ is $n-1$ dimensional and there are $m-1$ other planes so the maximum is $R(n-1, m-1)$. Therefore

$$R(n, m) = R(n, m-1) + R(n-1, m-1).$$

**Prop.**

$$R(n, m) = \sum_{i=0}^{n} \binom{m}{i}$$

**Proof.** We will use induction. We already have a recurrence so we need to just test the base cases. We just need to test the cases when $m = 0$ and $n = 1$. When $m = 0$, we will always get $R$ to be 1 which makes sense since when there are not any hyperplanes, we will divide our $n$ dimensional space into no pieces so the only region will be the $n$ dimensional space.

Now when $n = 1$, we get

$$R(1, m) = \sum_{i=0}^{1} \binom{m}{i} = m + 1.$$

This makes sense since $\mathbb{R}^1$ is just the number line and putting $m$ points on it splits the line into m+1 parts.

Therefore the maximum number of $r(\mathcal{A})$ is

$$\boxed{\sum_{i=0}^{n} \binom{m}{i}}.$$

# 6 Week 6

**(1)**

$$f\delta = \sum_{x \le z \le y} f(x, z)\delta(z, y) = f(x, y)\delta(y, y) = f$$

$$\delta f = \sum_{x \le z \le y} \delta(x,z) f(z,y) = \delta(x,x) f(x,y) = f$$

**(2)** We use the matrix representation of $f$ and $g$ and multiply them. Because we are not doing point wise multiplication, we can have two non-zero matricies and still have their product be the zero matrix. For example

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

**(3)** If we use the matrix representation of $f$ we can see that the following upper triangular matrix is idempotent:

$$\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

**(4)**

(a) We know that the total number of chains with no repeated elements in $B_n$ is

$$(2\delta - \zeta)^{-1}(x,y)$$

(b) The only chains of length 2 are ones where we have some subset and then the whole set. There are $\boxed{2^n}$ ways of picking the subset in the beginning so this is our answer.

**(5)** Since $\zeta(s)$ is a Dirichlet Series, we can use a Euler Product to expand

$$\zeta(s) = \prod_{\text{prime p}} \left( \frac{1}{1 - p^{-s}} \right) \implies \frac{1}{\zeta(s)} = \prod_{\text{prime p}} \left( 1 - \frac{1}{p^s} \right).$$

If we let $p_i$ bet the $i$th prime number starting from $p_1 = 2$, we have

$$\frac{1}{\zeta(s)} = \left(1 - \frac{1}{p_1^s}\right)\left(1 - \frac{1}{p_2^s}\right)\left(1 - \frac{1}{p_3^s}\right)\cdots.$$

Now we multiply everything out and group the terms based on how many prime there are in the denominator:

$$\frac{1}{\zeta(s)} = 1 - \left( \frac{1}{p_1^s} + \frac{1}{p_2^s} \cdots \right) + \left( \frac{1}{p_1^s p_2^s} + \frac{1}{p_1^s p_3^s} \cdots \frac{1}{p_2^s p_3^s} + \frac{1}{p_2^s p_4^s} + \cdots \right) - \cdots.$$

The denominator of this sum will be $m^s$ where $m$ is an integer with its prime factors having no power above 1. The signs are alternating so this will all be taken care of by $\mu(m)$. Therefore

$$\frac{1}{\zeta(s)} = 1 - \left( \frac{1}{p_1^s} + \frac{1}{p_2^s} \cdots \right) + \left( \frac{1}{p_1^s p_2^s} + \frac{1}{p_1^s p_3^s} \cdots \frac{1}{p_2^s p_3^s} + \frac{1}{p_2^s p_4^s} + \cdots \right) - \cdots = \sum_{m=0}^{\infty} \frac{\mu(m)}{m^s},$$

$$\zeta(s) \left( \sum_{m=0}^{\infty} \frac{\mu(m)}{m^s} \right) = \left( \sum_{m=0}^{\infty} \frac{\mu(m)}{m^s} \right) \left( \sum_{n=0}^{\infty} \frac{1}{n^s} \right) = 1.$$

**(6)** Let $p(n)$ be the number of primitive binary strings of length $n$. This is the function we want to find. There are a total of $2^n$ binary strings of length $n$. We can find this number in a different way by counting the number of primitive binary strings of length $d$ where $d|n$ and concatenating $n/d$ of them to form our binary string of length $n$. Then we sum over all $d$ so we get

$$2^n = \sum_{d|n} p(d).$$

Now we use the Möbius Inversion Formula on the poset $D_n$ to get

$$p(n) = \sum_{d|n} 2^d \mu(d, n).$$

Using our knowledge of the Möbius function on $D_n$ gives us

$$p(n) = \boxed{\sum_{d|n} 2^d \mu\left(\frac{n}{d}\right)}.$$

(We are now using the number theoretic Möbius Function).

**(7)**

$$\sum_{y\in P}\sum_{x\leq y} \mu(x, y) = \sum_{y\in P}\sum_{x\leq y} \frac{1}{\zeta(x, y)} = 1$$

# 7 Week 7

**(1)** We use the multinomial Theorem to get

$$(1 + x^3 + x^8)^{14} = \sum_{k_1+k_2+k_3} \binom{14}{k_1, k_2, k_3} x^{3k_2+8k_3}.$$

Then we see that the ordered pairs $(k_2, k_3)$ such that $3k_2 + 8k_3 = 30$ are $(2, 3)$ and $(10, 0)$. Therefore our coefficient of $x^{30}$ is

$$\binom{14}{8, 2, 3} + \binom{14}{4, 10, 0} = \boxed{182182}.$$

Here is the question this solves:

**Question.** You were part of Elon Musk's first Mars mission but due to some miscalculations you are now stranded on Mars. You have come upon Martian life and discovered that they play an interesting game with interesting dice. Their dice rolls either a 0, 3, or an 8 (of course they do not use Arabic Numerals). The goal of the game is to roll a 30. How many ways can you win?

**(2)** Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$. Now we multiply both sides of our recursion by $x^n$ and sum over $n = 0$ to $\infty$:

$$\sum_{n=0}^{\infty} a_{n+1}x^n = 2\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty}(3x)^n.$$

We can simplify the RHS into

$$2A(x) + \frac{1}{1 - 3x}$$

using the infinite geometric series formula. With a little bit of manipulation we can see that the LHS is

$$\frac{A(x) - a_0}{x} = \frac{A(x) - 2}{x}.$$

Therefore

$$\frac{A(x)-2}{x} = 2A(x) + \frac{1}{1-3x} \implies A(x) = \frac{2-5x}{(1-2x)(1-3x)} = \frac{1}{1-2x} + \frac{1}{1-3x}.$$

These two fractions are geometric series so therefore the coefficient of $x^n$ in $A(x)$ is

$$a_n = \boxed{2^n + 3^n}.$$

**(3)** Let $a_n = n^2$. Using this we get the recursion

$$a_0 = 0, a_{n+1} = a_n + 2n + 1.$$

Now we solve this recursion using generating functions.

Let $A(x) = \sum_{n=0}^{\infty} a_n x^n$. We do not solve the whole recursion but only solve for $A(x)$ and plug in $x = 1/2$. First we multiply both sides of the recursion by $x^n$ and sum from $n = 0$ to $\infty$:

$$\sum_{n=0}^{\infty} a_{n+1} x^n = \sum_{n=0}^{\infty} a_n x^n + 2\sum_{n=0}^{\infty} n x^n + \sum_{n=0}^{\infty} x^n.$$

The LHS is

$$\frac{A(x) - a_0}{x} = \frac{A(x)}{x}.$$

Now we find each sum in the RHS.

$$\sum_{n=0}^{\infty} a_n x^n = A(x),$$

$$\begin{aligned} 2\sum_{n=0}^{\infty} n x^n &= 2\sum_{n=1}^{\infty} (n-1) x^{n-1} \\ &= \frac{2}{x}\left(\sum_{n=0}^{\infty} n x^n - \sum_{n=1}^{\infty} x^n\right). \end{aligned}$$

Solving for $2\sum_{n=0}^{\infty} n x^n$ gives us

$$\left(1 - \frac{1}{x}\right) 2\sum_{n=0}^{\infty} n x^n = -\frac{2}{(1-x)} \implies 2\sum_{n=0}^{\infty} n x^n = \frac{2x}{(1-x)^2}.$$

Finally

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}.$$

Putting all this together gives us

$$\frac{A(x)}{x} = A(x) + \frac{2x}{(1-x)^2} + \frac{1}{1-x} = A(x) + \frac{1+x}{(1-x)^2}.$$

Solving for $A(x)$ gives us

$$A(x) = \frac{1+x}{(1-x)^2} \cdot \frac{1-x}{x} = \frac{1+x}{x(1-x)}.$$

Plugging in $x = 1/2$ gives us

$$A\left(\frac{1}{2}\right) = \boxed{6}$$

**(8)** The generating function of the partition function is

$$p(x) = (1 + x + x^2 \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^3 + x^6 + \cdots)\cdots$$

where each factor tells us what the part is and the exponent tells us how many of that part we have. Now we modify this power series to find the generating functions for the partitions with distinct parts and odd parts. We start with the former.

Distinct parts means that each each integer can be a part of the partition 0 or 1 times. Therefore each factor of our generating function will be in the form $1 + x^k$:

$$(1 + x)(1 + x^2)(1 + x^3)\cdots = \frac{1 - x^2}{1 - x} \cdot \frac{1 - x^4}{1 - x^2} \cdot \frac{1 - x^6}{1 - x^3} \cdots .$$

The numerators in the form $1 - x^k$ where $k$ is even will cancel with the denominator such that we will only be left with denominators in the form $1 - x^i$ where $i$ is odd:

$$\frac{1}{1 - x} \cdot \frac{1}{1 - x^3} \cdot \frac{1}{1 - x^5} \cdots .$$

For odd parts we will get

$$(1 + x + x^2 \cdots)(1 + x^3 + x^6 + \cdots)(1 + x^5 + x^{10} + \cdots)\cdots = \frac{1}{1 - x} \cdot \frac{1}{1 - x^3} \cdot \frac{1}{1 - x^5} \cdots .$$

This is exactly the same thing we got for the generating functions of partitions of distinct parts. Since the generating functions are equal, we are done.

**(9)** Similar to (8) we modify the generating function for the partition function to find the generating functions for the partitions where no part appears more than twice and partitions into parts that are not divisible by 3. We start with the former.

This means that each each integer can be a part of the partition 0, 1, 2 times. Therefore each factor of our generating function will be in the form $1 + x^k + x^{2k}$:

$$(1 + x + x^2)(1 + x^2 + x^4)(1 + x^3 + x^6)\cdots = \frac{1 - x^3}{1 - x} \cdot \frac{1 - x^6}{1 - x^2} \cdot \frac{1 - x^9}{1 - x^3} \cdots .$$

The numerators in the form $1 - x^k$ where $3|k$ will cancel with the denominator such that we will only be left with denominators in the form $1 - x^i$ where $i$ is not divisible by 3:

$$\prod_{3\nmid k} \frac{1}{1 - x^k} = \frac{1}{1 - x} \cdot \frac{1}{1 - x^2} \cdot \frac{1}{1 - x^4} \cdots .$$

For partitions with no parts divisible by 3 we will get

$$(1 + x + x^2 \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^4 + x^8 + \cdots)\cdots = \frac{1}{1 - x} \cdot \frac{1}{1 - x^2} \cdot \frac{1}{1 - x^4} \cdots .$$

This is exactly the same thing we got for the generating functions of partitions of distinct parts. Since the generating functions are equal, we are done.

We can generalize the results from (8) and (9):

**Prop.** The number of partitions that have parts not divisible by $k$ are the same as the number of partitions where no part can appear more that $k-1$ times.

The proof of this is essentially the same as the proofs of (8) and (9).

**(10)** The proof to this is basically the same as the previous few problems but we will have different generating functions that are equal.

**(11)** We do this like the last few problems we have proved. Even parts that are distinct has the generating function

$$(1+x^2)(1+x^4)(1+x^6)\cdots = \frac{1-x^4}{1-x^2}\cdot\frac{1-x^8}{1-x^2}\cdot\frac{1-x^{12}}{1-x^2}\cdots.$$

The numerators are all in the form $1-x^k$ where $4|k$ and this will cancel with the denominator and we get

$$\prod_{4\nmid k}\frac{1}{1-x^k}.$$

This is the generating function for partitions that are not divisible by 4. Now by 1.2, we have that the generating function also represents partitions where no part appears more than 3 times. Therefore we are done.

**(12)** Let us first find a recurrence for $f$. If $n$ is not in our subset, we have $f(n-1)$ subsets since we can pick from $n-1$ numbers. If $n$ is in our subset, we have $f(n-4)$ subsets since we pick from $n-4$ numbers. Therefore

$$f(n) = f(n-1) + f(n-4) \implies f(n+4) = f(n+3) + f(n)$$

with initial conditions $f(0)=1$, $f(1)=2$, $f(2)=3$, and $f(3)=4$. Now we solve this recursion using generating functions. (We don't fully solve for a closed form but we only solve for the generating function of the recursion). Now we solve this problem similar to (3).

**(13)** Let $a_n$ be the number of ways of placing the dominoes on a $2\times n$ board. Now we form a recursion. We can either start with one vertical dominoes and tile the rest in $a_{n-1}$ ways or we can start with two horizontal dominoes and tile the rest in $a_{n-2}$ ways. Therefore

$$a_n = a_{n-1} + a_{n-2}, a_1 = 1.$$

Therefore

$$a_n = \boxed{F_n}$$

**(18)** Let $k > 4$ be any integer. Therefore the only bigger rulers that have optimal markings are rulers of length

$$\binom{k}{2}.$$

The way we construct the bigger ruler is by having the same intervals as the length 6 ruler but we have to add $k-4$ more numbers. We can see that no matter how we assign values to the $k-4$ places, we will never be able to create a bigger ruler.

## 8 Week 8

**(1)**

(a) We can think of the $a$'s as rights and $b$'s as ups and we get Dyck Paths.

(b) We create a bijection with the parenthesis representation of the Catalan numbers. Here is how we do it: first label the sides $a_i$ for $i = 1, 2, \cdots, n+2$ for each diagonal, we look at its two points. If there is a diagonal coming out of one of the points, we consider it. If not, we consider the side coming out of the point. Then we concatenate what we get from both sides and put parenthesis around it. We can then notice that the the total is the $n$th catalan number.

**(2)** Let $L_n$ be the number of binary trees with $n$ leaves. We must show that $L_{n+1}$ satisfies the Catalan recurrence. We can split each binary tree into two subtrees: one where the left child of the root is the root of the subtree and one where the right child of the root is the root of the subtree. Now let there be $k$ leaves in the left subtree and $n-k$. By definition, there are $L_k$ ways of creating the left subtree and $L_{n-k}$ ways of creating the right subtree. This gives us a total of $L_kL_{n-k}$ trees where the left subtree has $k$ leaves. Summing over all $k$ gives us

$$L_n = \sum_{k=0}^{n} L_k L_{n-k}.$$

Therefore we are done.

**(3)** We have

$$\begin{aligned}\frac{1}{1-(x+y)} &= 1 + (x+y) + (x+y)^2 + (x+y)^3 + \cdots \\ &= \sum_{m=0}^{\infty} (x+y)^m\end{aligned}$$

Now by the Binomial Theorem, we get what we want

$$\sum_{m=0}^{\infty} (x+y)^m = \sum_{m=0}^{\infty} \sum_{n=0}^{m} \binom{m}{n} x^n y^{m-n}$$

**(6)** Let there be $a_n$ binary sequences of length $n$ with no adjacent 1s. The first two digits can be 00, 10, or 01. For the first two cases, we can construct $a_{n-2}$ sequences. For the third case, the third digit cannot be a 1 so there are $a_{n-3}$ sequences. Therefore we have the recursion

$$a_n = 2a_{n-2} + a_{n-3}.$$

**(7)** Similar to (6), we get the recurrence

$$a_n = 2a_{n-1} + a_{n-2}$$

# 9 Week 9

**(1)**

$$\begin{aligned}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!}\right)\left(\sum_{n=0}^{\infty}\frac{y^n}{n!}\right) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{x^n y^m}{m!n!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{x^n y^{n-m}}{(n-m)!n!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}\frac{x^n y^{n-m}}{n!} \\ &= \sum_{n=0}^{\infty}\frac{(x+y)^n}{n!}\end{aligned}$$

**(2)**

$$\begin{aligned}\sum_{n=0}^{\infty} a_{n+1}\frac{x^n}{n!} &= \frac{n+1}{x}\left(\sum_{n=0}^{\infty}\left(a_n\frac{x^n}{n!}\right) - a_0\right) \\ &= \boxed{\frac{(n+1)(A(x)-a_0)}{x}}\end{aligned}$$

**(3)**

$$\begin{aligned}\zeta(s)^k &= \sum_{a_1=1}^{\infty}\sum_{a_2=1}^{\infty}\cdots\sum_{a_n=1}^{\infty}\frac{1}{(a_1a_2\cdots a_n)^s} \\ &= \sum_{n=1}^{\infty}\frac{N}{n^s}\end{aligned}$$

Where $N$ is the number of ways of writing $n$ as the product of $k$ natural numbers.

**(4)** Since $\zeta(s)$ is a Dirichlet Series, we can use a Euler Product to expand

$$\zeta(s) = \prod_{\text{prime p}}\left(\frac{1}{1-p^{-s}}\right) \implies \frac{1}{\zeta(s)} = \prod_{\text{prime p}}\left(1-\frac{1}{p^s}\right).$$

If we let $p_i$ bet the $i$th prime number starting from $p_1 = 2$, we have

$$\frac{1}{\zeta(s)} = \left(1-\frac{1}{p_1^s}\right)\left(1-\frac{1}{p_2^s}\right)\left(1-\frac{1}{p_3^s}\right)\cdots.$$

Now we multiply everything out and group the terms based on how many prime there are in the denominator:

$$\frac{1}{\zeta(s)} = 1-\left(\frac{1}{p_1^s}+\frac{1}{p_2^s}\cdots\right)+\left(\frac{1}{p_1^sp_2^s}+\frac{1}{p_1^sp_3^s}\cdots\frac{1}{p_2^sp_3^s}+\frac{1}{p_2^sp_4^s}+\cdots\right)-\cdots.$$

The denominator of this sum will be $m^s$ where $m$ is an integer with its prime factors having no power above 1. The signs are alternating so this will all be taken care of by $\mu(m)$. Therefore

$$\frac{1}{\zeta(s)} = 1-\left(\frac{1}{p_1^s}+\frac{1}{p_2^s}\cdots\right)+\left(\frac{1}{p_1^sp_2^s}+\frac{1}{p_1^sp_3^s}\cdots\frac{1}{p_2^sp_3^s}+\frac{1}{p_2^sp_4^s}+\cdots\right)-\cdots = \sum_{m=0}^{\infty}\frac{\mu(m)}{m^s}.$$

**(5)**

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{b_n}{n^s} &= \sum_{n=1}^{\infty}\frac{\sum_{d|n} a_d}{n^s}\\ &= \sum_{n=1}^{\infty}\sum_{d|n}\frac{a_d}{n^s}\\ &= \frac{a_1}{1^s}+\left(\frac{a_1}{2^s}+\frac{a_2}{2^s}\right)+\left(\frac{a_1}{3^s}+\frac{a_3}{3^s}\right)+\cdots\\ &= a_1\left(\frac{1}{1^s}+\frac{1}{2^s}\cdots\right)+a_2\left(\frac{1}{2^s}+\frac{1}{4^s}\cdots\right)\cdots\\ &= \frac{a_1}{1^s}\left(\frac{1}{1^s}+\frac{1}{2^s}\cdots\right)+\frac{a_2}{2^s}\left(\frac{1}{1^s}+\frac{1}{2^s}\cdots\right)\cdots\\ &= \left(\frac{a_1}{1^s}+\frac{a_2}{2^s}\cdots\right)\left(\frac{1}{1^s}+\frac{1}{2^s}\cdots\right)\\ &= \boxed{A(s)\zeta(s)}\end{aligned}$$

**(6)** Let $f(x)=\sum_{n=0}^{\infty}F_n\frac{x^n}{n!}$. Now we can see that

$$\sum_{n=0}^{\infty}F_{n+2}\frac{x^n}{n!}=\sum_{n=0}^{\infty}F_{n+1}\frac{x^n}{n!}+\sum_{n=0}^{\infty}F_n\frac{x^n}{n!}\implies\frac{d^2f}{dx^2}=\frac{df}{dx}+f.$$

Our characteristic polynomial for this equation is

$$r^2-r-1.$$

Our roots are $r=\frac{1\pm\sqrt{5}}{2}$. Therefore our solutions are

$$f(x)=e^{x(1+\sqrt{5})/2}$$

and

$$f(x)=e^{x(1-\sqrt{5})/2}.$$

Now we use the Principle of Superposition to get

$$f(x)=c_1e^{x(1+\sqrt{5})/2}+c_2e^{x(1-\sqrt{5})/2}.$$

we can make this simpler by letting $\phi_1=\frac{1+\sqrt{5}}{2}$ and $\phi_2=\frac{1-\sqrt{5}}{2}$:

$$f(x)=c_1e^{x\phi_1}+c_2e^{x\phi_2}$$

We can find the constants by using the fact that $f(0)=0$ and $f'(0)=1$ to get

$$f(x)=\boxed{\frac{e^{x\phi_1}-e^{x\phi_2}}{\sqrt{5}}}.$$

Using the Maclaurin Series of $e^x$ gives us

$$\frac{1}{\sqrt{5}}\left(\sum_{n=0}^{\infty}\frac{(\phi_1x)^n}{n!}+\sum_{n=0}^{\infty}\frac{(\phi_2x)^n}{n!}\right)=\left(\sum_{n=0}^{\infty}\frac{\phi_1^n+\phi_2^n}{\sqrt{5}}\cdot\frac{x^n}{n!}\right).$$

Therefore

$$F_n=\frac{\phi_1^n+\phi_2^n}{\sqrt{5}}=\boxed{\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n+\left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}}.$$