AN INTRODUCTION TO DE RHAM COHOMOLOGY

NANDANA MADHUKARA

Contents

1. Introduction

In the realm of differential geometry and topology, the study of differential forms provides a powerful framework for understanding the geometric properties of manifolds and their associated tangent vector fields. From the intuitive idea of measuring tangent vectors to the profound insights of curvature and integration, the language of forms allows us to investigate the deepest aspects of geometry. At the heart of this exploration lies a remarkable mathematical concept known as De Rham cohomology.

De Rham cohomology is a mathematical tool that captures the essential topological information of a manifold through its differential forms. Developed by the Swiss mathematician Georges de Rham in the mid-20th century, this branch of mathematics has since become a cornerstone of modern differential geometry and topology. It provides a powerful language to classify and understand the topological structure of manifolds by examining the behavior of differential forms under the operations of differentiation and integration.

The primary objective of this expository paper is to guide through the captivating world of De Rham cohomology. We will embark on a journey that takes us from the basic concepts of differential forms to the intricacies of cohomology theory, unraveling its significance and applications along the way. By delving into the essential ideas and techniques, we will gain a comprehensive understanding of how De Rham cohomology captures the topological essence of a manifold.

This paper will first start off with background results from exterior algebra and exterior calculus that will aid us in our discussion about De Rham cohomology. Then we will actually define these cohomology groups and prove the remarkable property that they are homotopy invarient. In this paper, we will talk a lot about manifolds so when we do so, we are actually mean smooth manifolds.

Date: May 28, 2024.

2. Exterior Alegbra

Before we get into cohomolgies, we need some background in differential forms which starts with exterior algebra.

Definition 2.1 (Tensor)**.** A p-tensor on a vector space V is any real-valued function T such that on V^p that is multilinear i.e.

$$T(v_1, ..., v_j + av_j', ..., v_p) = T(v_i, ..., v_j, ..., v_p) + aT(v_i, ..., v_j', ..., v_p).$$

We call the collection of all p-tensors $\mathcal{J}^p(V^*)$.

Tensors are like measurements we can take of vectors in some vector space since it assigns a set of vectors to a real value.

Example. We can see that a 1-tensor is $T : V \to \mathbb{R}$ so it is a linear form. This means that $\mathcal{J}^p(V^*) = V^*$ or the dual space of V. Additionally, there is also a familiar 2-tensor which is the dot product. This is like a measurement of how orthogonal two vectors are. We can also measure the volume of the parallelipiped formed by the vectors using the determinant which is a p-tensor on $\mathbb{R}^p$. Specifically, this tensor is defined by

$$T(v_1, ..., v_p) = \det \begin{pmatrix} v_1 & \cdots & v_p \end{pmatrix}$$

and we know that this determinant is multilinear.

We can also multiply tensors:

Definition 2.2 (Tensor Product)**.** If T is a p-tensor and S is a q-tensor, then $T \otimes S$ is a $p + q$ tensor defined by

$$T \otimes S(v_1, ..., v_p, u_1, ..., u_q) = T(v_1, ..., v_p) \cdot S(u_1, ..., u_q)$$

Notice that tensor products are not commutative and

$$T \otimes S \neq S \otimes T.$$

Next we introduce alternating tensors. Let S_p be the set of permuations of $\{1, 2, ..., p\}$ and let $\pi \in S_p$ be a permutation. Now notice that π can be decomposed into a series of swaps of two elements of the set. We can assign the parity of π as the parity of the number of swaps π decomposes into. Then we can write $(-1)^\pi$ which is $+1$ when π is even and -1 when π is odd.

Definition 2.3 (Alternating Tensor)**.** A p-tensor T is called *alternating* if

$$T = (-1)^\pi T^\pi$$

where

$$T^\pi(v_1, v_2, ..., v_p) = T(v_{\pi(1)}, v_{\pi(2)}, ..., v_{\pi(p)})$$

Intuitively, this means that every time we swap to vectors, the sign of the tensor switches. Conveniently, the sum and scalar multiples of alternating tensors are still alternating tensors so the set of all alternating tensors which we call $\Lambda^p(V^*)$ is a subspace of $\mathcal{J}^p(V^*)$. Unfortunately, the tensor product of two alternating tensors does not produce another alternating tensor so for this, we introduce the wedge product. First we define how we can construct alternating tensors:

Definition 2.4. Let T be a p-tensor. We define the function $\operatorname{Alt}(T)$ as

$$\operatorname{Alt}(T) = \frac{1}{p!} \sum_{\pi \in S_p} (-1)^\pi T^\pi.$$

Let us indeed verify that $\operatorname{Alt}(T)$ produces an alternating tensor.

Proposition 2.5. *If T is a p-tensor, the tensor $\operatorname{Alt}(T)$ is alternating.*

Proof. Let $\sigma \in S_p$ be a permutation. We have

$$[\operatorname{Alt}(T)]^\sigma = \frac{1}{p!} \sum_{\pi \in S_p} (-1)^\pi (T^\pi)^\sigma.$$

Now notice that $(-1)^{\pi \circ \sigma}$ is $+1$ when π and σ have the same parity and -1 when π and σ have different parities. Therefore, we see that

$$(-1)^{\pi \circ \sigma} = (-1)^\pi (-1)^\sigma \implies (-1)^\pi = (-1)^{\pi \circ \sigma} (-1)^\sigma.$$

Plugging this in gives us

$$[\operatorname{Alt}(T)]^\sigma = (-1)^\sigma \frac{1}{p!} \sum_{\pi \in S_p} (-1)^{\pi \circ \sigma} T^{\pi \circ \sigma}.$$

Next we let $\tau = \pi \circ \sigma$. Since S_p is a group under composition, when π spans S_p, so does τ. Therefore,

$$[\operatorname{Alt}(T)]^\sigma = (-1)^\sigma \frac{1}{p!} \sum_{\tau \in S_p} (-1)^\tau T^\tau = (-1)^\sigma \operatorname{Alt}(T)$$

which proves that $\operatorname{Alt}(T)$ is alternating. ∎

Example. If T is a 1-tensor, we must check if $(-1)^\pi T^\pi = T$ for every $\pi \in S_1$ but this is clearly true since π is the identity permutation so it is even and $T^\pi = T$. This shows us that all 1-tensors or linear forms are alternating.

Example. When T is alternating,

$$\operatorname{Alt}(T) = \frac{1}{p!} \sum_{\pi \in S_p} (-1)^\pi T^\pi = \frac{1}{p!} \sum_{\pi \in S_p} T = T$$

where we are using the fact that $T = (-1)^\pi T^\pi$. This is why we have a factor of $1/p!$ in the front.

Definition 2.6 (Wedge Product)**.** If $T \in \Lambda^p(V^*)$ and $S \in \Lambda^q(V^*)$, the *wedge product* is the $p+q$ tensor $T \wedge S \in \Lambda^{p+q}(V^*)$ defined by

$$T \wedge S = \operatorname{Alt}(T \otimes S).$$

Clearly, this product distributes over addition and scalar multiplication since Alt is a linear operation. Additionally, with some work, we can also see that the wedge product is associative.

This allows us to derive a basis for $\Lambda^p(V^*)$. If T is a p-tensor, then we can write

$$T = \sum_I t_{i_1,\ldots,i_k} \phi_{i_1} \otimes \cdots \otimes \phi_{i_p}$$

where $\{\phi_1, ..., \phi_k\}$ is a basis of V^* and the sum ranges over all index sequences $I = (i_1, ..., i_p)$ with $1 \leq i_1, ..., i_p \leq k$. If T is alternating, then

$$T = \mathrm{Alt}(T) = \sum_I t_{i_1,\ldots,i_p} \mathrm{Alt}(\phi_{i_1} \otimes \cdots \otimes \phi_{i_p}) = \sum_I t_{i_1,\ldots,i_p} \phi_{i_1} \wedge \cdots \wedge \phi_{i_p}.$$

We denote $\phi_{i_1} \wedge \cdots \wedge \phi_{i_k}$ by ϕ_I. We have shown that the ϕ_I's span $\Lambda^p(V^*)$ but they are not linearly independent.

Lemma 2.7. *The set* $\{\phi_I : 1 \leq i_1 < i_2 < \cdots < i_p \leq k\}$ *is linearly independent.*

Proof. Let $\{v_1, ..., v_k\}$ be the basis of V dual to $\{\phi_1, ..., \phi_k\}$. This means that

$$\phi_i(v_j) = \delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}.$$

For any increasing index sequence $I = (i_1, ..., i_p)$, we let $v_I = (v_{i_1}, ..., v_{i_p})$. We see

$$\phi_I(v_I) = \frac{1}{p!} \sum_{\pi \in S_p} (-1)^\pi (\phi_{i_1} \otimes \cdots \otimes \phi_{i_p})^\pi (v_I) = \frac{1}{p!} \sum_{\pi \in S_p} (-1)^\pi \phi_{i_1}(v_{\pi(i_1)}) \cdots \phi_{i_p}(v_{\pi(i_p)}).$$

Notice that the product inside the sum is 0 unless π is the identity permutation where the product is 1. This is because if $\pi(i_n) \neq i_n$ for $n \in \{1, 2, ..., p\}$, then $\phi(v_{\pi(i_n)}) = 0$. Therefore, this shows us that $\phi_I(v_I) = 1/p!$. Now if J is a different increasing index sequence than I, there is always going to be at least one value of n such that $\pi(j_n) \neq i_n$ so $\phi_I(v_J) = 0$. Therefore, if $\sum_I a_I \phi_I = 0$, for any J, we know that

$$0 = \sum_I a_I \phi(v_J) = \frac{1}{p!} a_J \implies a_J = 0.$$

Since all the weights are 0, we have proved that $\{\phi_I\}$ is linearly independent. ■

Putting our discussion together gives us following result:

Proposition 2.8. *If* $\{\phi_1, \phi_2, ..., \phi_k\}$ *is a basis of* V^**, then* $\{\phi_I : 1 \leq i_1 < i_2 < \cdots < i_p \leq k\}$ *is a basis of* $\Lambda^p(V^*)$*. Additionally,*

$$\dim \Lambda^p(V^*) = \binom{k}{p}.$$

Finally, consider two linear forms $\phi, \psi \in \Lambda^1(V^*)$. We have

$$\phi \wedge \psi = \mathrm{Alt}(\phi \otimes \psi) = \frac{1}{2} \sum_{\pi \in S_2} (-1)^\pi (\phi \otimes \psi)^\pi = \frac{1}{2}(\phi \otimes \psi - \psi \otimes \phi).$$

This means that $\phi \wedge \psi = -\psi \wedge \phi$ so the wedge product is anticommutative. Additionally, we can see that $\phi \wedge \phi = 0$. It can be seen that this extends to

$$\phi_I \wedge \phi_J = (-1)^{pq} \phi_J \wedge \phi_I$$

so by Proposition 2.8, we have

$$T \wedge S = (-1)^{pq} S \wedge T$$

where T and S are p and q alternating tensors, respectively.

3. Exterior Calculus

So far, we have generalized linear algebra from talking about one dimensional vectors to talking about "p dimensional" vectors. Just like in vector calculus, we can do calculus on this generalized version linear algebra and this is called exterior calculus.

Definition 3.1 (Differential p-forms)**.** Let X be a manifold. A differential p-form (or a p-form for short) on X is a function ω that assigns each point $x \in X$ to an alternating p-tensor ω_x on the tangent space of X at x. This means that $\omega_x \in \Lambda^p(T_x(X)^*)$. The set of all p-forms on X is denoted by $\Omega^p(X)$.

Recall that tensors are like measuring devices for a vector space. Now in a differential form, we assign each point on a manifold to these measuring devices so a differential form tells us how to measure tangent vectors at each point on a manifold.

We can do operations on these forms. We can add two p-forms:

$$(\omega + \omega')_x = \omega_x + \omega'_x.$$

We can also take the wedge product of a p and q form:

$$(\omega \wedge \theta)_x = \omega_x \wedge \theta_x.$$

Anticommutativity follows from the result from before

$$\omega \wedge \theta = (-1)^{pq}\theta \wedge \omega.$$

Example. A 0-form is a function that assigns each point $x \in X$ to an alternating 0-tensor which is just a real value. Therefore a 0-form is just a real-valued function on X.

Example. Now a 1-form is a function that assigns each point $x \in X$ to an alternating 1-tensor on $T_x(X)$. If $\phi : X \to \mathbb{R}$ is a smooth real-valued function, then $d\phi_x : T_x(X) \to \mathbb{R}$ is a 1-tensor on $T_x(X)$. This is alternating since all 1-tensors are alternating. Therefore the mapping $x \mapsto d\phi_x$ defines a 1-form $d\phi$ on X called the *differential* of ϕ.

Consider the coordinate functions $x_1, x_2, ..., x_k$ of $\mathbb{R}^k$ that map a vector to its ith coordinate. These coordinate functions give us the differentials $dx_1, dx_2, ..., dx_k$ so for any $z \in \mathbb{R}^k$, the linear forms $(dx_1)_z, (dx_2)_z, ..., (dx_k)_z$ are just the standard basis of $(\mathbb{R}^k)^*$. Let U be some open subset of $\mathbb{R}^k$ containing z. By Proposition 2.8, the set $\{(dx_I)_z\}$ where I is an increasing index sequence and

$$(dx_I)_z = (dx_{i_1})_z \wedge \cdots \wedge (dx_{i_p})_z$$

is a basis of $\Lambda^p((R^k)^*) = \Lambda^p(T_z(U)^*)$. This means that any alternating tensor $\omega(z)$ on $T_z(U)$ can be uniquely written as

$$\sum_I a_I(dx_I)_z = \sum_I a_I(z)(dx_I)_z$$

where the a_I's are real-valued functions on U. This gives us the following result:

Proposition 3.2. *Every p-form on an open set $U \subseteq \mathbb{R}^k$ can be uniquely written as $\sum_I a_I dx_I$ where the sum ranges over increasing index sequences and the a_I are real-valued functions or 0-forms on U.*

This allows us to take the derivative of differential forms.

Definition 3.3 (Exterior Derivative)**.** Let $\omega = \sum_I a_I dx_I$ be a smooth p-form on an open set of $\mathbb{R}^k$. The exterior derivative of ω is the $(p+1)$-form

$$d\omega = \sum_I da_I \wedge dx_I.$$

We can generalize this definitions to any manifold X by converting a local neighborhood of $x \in X$ in Euclidean space with a coordinate chart.

Example. Let $\omega_1 = \sum_I a_I dx_I$ and $\omega_2 = \sum_I b_I dx_I$ be two p-forms. Since $\omega_1 + \omega_2 = \sum_I (a_I + b_I) dx_I$, we have

$$\begin{aligned} d(\omega_1 + \omega_2) &= \sum_I d(a_I + b_I) \wedge dx_I \\ &= \sum_I (da_I + db_I) \wedge dx_I \\ &= \sum_I da_I \wedge dx_I + \sum_I db_I \wedge dx_I \\ &= d\omega_I + d\omega_2. \end{aligned}$$

which is the sum rule for exterior derivatives.

Example. Let $\omega = \sum_I a_I dx_I$ be a p-form and let $\theta = \sum_J b_J dx_J$ be a q form. Let us compute $d(\omega \wedge \theta)$ to derive the prodcut rule. We have,

$$d(\omega \wedge \theta) = d\left(\left(\sum_I a_I dx_I\right) \wedge \left(\sum_J b_J dx_J\right)\right) = \sum_I \sum_J d(a_I dx_I \wedge b_J dx_J).$$

(We have implicitly used the sum rule when we brought the derivative to the inside.) Now we derive the derivative inside the sum. Let $\omega'_I = a_I dx_I$ and $\theta'_J = b_J dx_J$. We have

$$\begin{aligned} d(a_I dx_I \wedge b_J dx_J) &= d(a_I b_J) \wedge dx_I \wedge dx_J \\ &= (b_j\, da_I + a_I\, db_J) \wedge dx_I \wedge dx_J \\ &= b_J\, da_I \wedge dx_I \wedge dx_J + a_I\, db_J \wedge dx_I \wedge dx_J \\ &= (da_I \wedge dx_I) \wedge (b_J \wedge dx_J) + (-1)^p (a_I \wedge dx_I) \wedge (db_J \wedge dx_J) \\ &= d\omega'_I \wedge \theta'_J + (-1)^p \omega'_I \wedge d\theta'_J. \end{aligned}$$

Notice that the reason $(-1)^p$ arrises is because we switch the order of dx_I, which is a p-form, and db_J, which is a 1-form. Plugging this back into the sum gives us

$$\begin{aligned} d(\omega \wedge \theta) &= \sum_I \sum_J d(a_I dx_I \wedge b_J dx_J) \\ &= \sum_I \sum_J d\omega'_I \wedge \theta'_J + (-1)^p \omega'_I \wedge d\theta'_J \\ &= \sum_I \sum_J d\omega'_I \wedge \theta'_J + (-1)^p \sum_I \sum_J \omega'_I \wedge d\theta'_J \\ &= \left(\sum_I d\omega'_I\right) \wedge \left(\sum_J \theta'_J\right) + (-1)^p \left(\sum_I \omega'_I\right) \wedge \left(\sum_J d\theta'_J\right) \\ &= d\omega \wedge \theta + (-1)^p \omega \wedge d\theta \end{aligned}$$

which is the product rule.

Example. Let $\theta = \sum_I a_I dx_I$. This means that

$$d\theta = \sum_I da_I \wedge dx_I = \sum_I \left(\sum_{i=1}^k \frac{\partial a_I}{\partial x_i} dx_i \right) \wedge dx_I.$$

This gives us

$$\begin{aligned} d(d\theta) &= \sum_I d\left(\sum_{i=1}^k \frac{\partial a_I}{\partial x_i} dx_i \right) \wedge dx_I - \left(\sum_{i=1}^k \frac{\partial a_I}{\partial x_i} dx_i \right) \wedge d(dx_I) \\ &= \sum_I d\left(\sum_{i=1}^k \frac{\partial a_I}{\partial x_i} dx_i \right) \wedge dx_I \\ &= \sum_I \sum_{i=1}^k d\left(\frac{\partial a_I}{\partial x_i} \right) \wedge dx_i \wedge dx_I \\ &= \sum_I \sum_{i=1}^k \left(\sum_{j=1}^k \frac{\partial^2 a_I}{\partial x_i \partial x_j} dx_j \right) \wedge dx_i \wedge dx_I \\ &= \sum_I \left(\sum_{i=1}^k \sum_{j=1}^k \frac{\partial^2 a_I}{\partial x_i \partial x_j} dx_j \wedge dx_i \right) \wedge dx_I \end{aligned}$$

by the product rule and since the derivative of dx_I is 0. Since

$$\frac{\partial^2 a_I}{\partial x_i \partial x_j} = \frac{\partial^2 a_I}{\partial x_j \partial x_i}$$

and $dx_i \wedge dx_j = -dx_j \wedge dx_i$, the terms in the parenthesized sum cancel out so the sum is 0. This is called the *cocycle condition.*

So far, we have been discussing p-forms on a single manifold but we can map p-forms from one manifold to another.

Definition 3.4 (Pullback Map)**.** If $f : X \to Y$ be a smooth map and let $df_x : T_x(X) \to T_{f(x)}(Y)$ be the derivative. The linear map $f^*\omega : \Omega^p(Y) \to \Omega^p(X)$ is called the *pullback by f at x* and maps a p-form on Y, ω, to a p-form on x, $f^*\omega$, defined by

$$(f^*\omega)_x(v_1, ..., v_k) = \omega_{f(x)}(df_x(v_1), ..., df_x(v_k)).$$

One of the most important properties of the pullback

Proposition 3.5. *The pullback commutes with the exterior derivative.*

We will not prove this result in this paper but it can be seen by these other properties of the pullback map:

Proposition 3.6. *Let $f : X \to Y$ be a smooth map.*

(a) f^* *is linear over all* $\mathbb{R}$
(b) $f^*(\omega \wedge \theta) = f^*\omega \wedge f^*\theta$

(c) *In any smooth chart on* Y,

$$f^*\left(\sum_I a_I dy_I\right) = \sum_I (a_I \circ f) d(y_{i_1} \circ f) \wedge \cdots \wedge d(y_{i_p} \circ f).$$

4. De Rham Cohomology Groups

Now we have built up the background necessary to talk about De Rham cohomolgies.

Definition 4.1. A p-form ω on X is *closed* if $d\omega = 0$ and *exact* if $\omega = d\theta$ for some $(p-1)$-form θ. The set of all closed p-forms on X is denoted by $Z^p(X)$ and the set of all exact p-forms on X is denoted by $B^p(X)$

Proposition 4.2. *All exact forms are closed.*

Proof. If ω is exact, then $d\omega = d(d\theta)$. By the cocycle condition, we have $d(d\theta) = d\omega = 0$ so ω is closed. ∎

Notice that the reverse does not hold i.e. closed $\not\Longrightarrow$ exact. In order to investigate this further, we define an equivalence relation of closed p-forms. We call two closed p forms ω_1, ω_2 *cohomologous* if $\omega_1 - \omega_2$ is exact and this is denoted by $\omega_1 \sim \omega_2$. The set of equivalence classes is the De Rham cohomology group:

Definition 4.3 (De Rham Cohomology Group)**.** Consider the following sequence

$$0 \to \Omega^0 \xrightarrow{d^0} \Omega^1 \xrightarrow{d^1} \Omega^2 \xrightarrow{d^2} \cdots$$

where d^p is the exterior derivative on p-forms. The pth *De Rham cohomology group* (or pth *cohomology group* for short) is $H^p(X) = \ker(d^p)/\operatorname{im}(d^{p-1})$.

An element of $H^p(X)$ is called a *cohomology class* and the cohomology class containing the p-form ω is denoted by $[\omega]$ i.e.

$$[\omega] = \{\omega + d^{p-1}\omega' : \omega' \in \Omega^{p-1}\}.$$

First, we know that $H^p(X)$ is well defined since $d^p(d^{p-1}(\omega)) = 0$ so $\operatorname{im}(d^{p-1}) \subseteq \ker(d^p)$. Second, notice that $\ker(d^p)$ is the same as $Z^p(X)$ and $\operatorname{im}(d^{p-1})$ is the same as $B^p(X)$. This why $H^p(X) = Z^p(X)/\sim$ (which is what we said before the definition) since two closed p-forms ω_1, ω_2 will be in the same coset iff $\omega_1 - \omega_2 \in \operatorname{im}(d^{p-1}) \implies \omega_1 \sim \omega_2$.

The pth De Rham cohomology group is much more than just a set of cosets: it has a natural vector space structure. Notice that if $\omega_1 \sim \omega_1'$ and $\omega_2 \sim \omega_2'$, then $\omega_1 + \omega_2 \sim \omega_1' + \omega_2'$. Additionally, we have $c\omega \sim c\omega_1$. Therefore, the normal vector operations on closed p-forms can be extended to the elements of $H^p(X)$.

5. Homotopy Invarience

Now one of the amazing things about the pth cohomology group is that it is invarient to homotopies. That is, if we deform X, its cohomology groups will remain the same (or more specifically be isomorphic to the original). Before we get into this, we must prove preleminary results.

Proposition 5.1. *Let* $f : X \to Y$ *between a smooth map. The pullback* $f^* : \Omega^p(Y) \to \Omega^p(X)$ *carries* $Z^p(Y)$ *into* $Z^p(X)$ *and* $B^p(Y)$ *into* $B^p(X)$.

Proof. If ω is closed, then $d(f^*\omega) = f^*(d\omega) = 0$ so $f^*\omega$ is closed. If $\omega = d\theta$ is exact, then $f^*\omega = f^*(d\theta) = d(f^*\theta)$ so $f^*\omega$ is exact. ∎

Definition 5.2. If $f : X \to Y$ is a smooth map, the pullback f^* induces a linear map from $H^p(Y)$ to $H^p(X)$, still denoted by f^*, (because of Proposition 5.1) defined naturally as

$$f^*[\omega] = [f^*\omega]$$

and this is called the *induced cohomology map.*

Now if $\omega' \sim \omega \implies \omega' = \omega + d\theta$,

$$f^*\omega' = [f^*\omega'] = [f^*\omega + d(f * \theta)] = [f^*\omega]$$

so the map from the definition above is well-defined.

Proposition 5.3. *Let $f : X \to Y$ be a smooth map. The induced cohomology map follows the these properties:*

(a) *If $g : Y \to Z$ is another smooth map, then*

$$(g \circ f)^* = f^* \circ g^*.$$

(b) $\mathrm{Id}_X^* = \mathrm{Id}_{H^p(X)}$.

Proof. ∎

Now we are ready to discuss homotopy invarience of cohomology groups. The underlying fact behind the proof is that homotopic smooth maps induce the same cohomology map. Let us look at what $f^* = g^*$ means for $f, g : X \to Y$. If $\omega \in \Omega^p(Y)$, we need to find a $(p-1)$-form θ such that

$$f^*\omega - g^*\omega = d\theta$$

which would mean that $f^*[\omega] - g^*[\omega] = [f^*\omega] - [g^*\omega] = [d\theta] = 0$ so the two cosets are the same. We can produce this θ with a linear map $h : Z^p(Y) \to Z^{p-1}(X)$ which gives us

$$f^*\omega - g^*\omega = d(h\omega).$$

We can generalize h as a map from $\Omega^p(Y)$ to $\Omega^{p-1}(X)$ but our condition for $f^* = g^*$ changes to

$$f^*\omega - g^*\omega = d(h\omega) + h(d\omega).$$

Notice that when ω is closed, we have $d\omega = 0$ so the above condition reduces to our original condition. To summarize, if there exists a linear map h such that the above condition is satisfied, then $f^* = g^*$. The map h is called a homotopy operator between f^* and g^*.

If X is a manifold and $t \in I$, let $i_t : X \to X \times I$ be the map

$$i_t(x) = (x, t).$$

(Notice that we are using the symbols i_t and I again so depending on the context, they will mean different things.) They key to proving homotopy invarience is to construct a homotopy operator between i_0^* and i_1^* to prove that they are equal.

Lemma 5.4. *For any manifold X, there exists a homotopy operator between i_0^* and i_1^* for every p.*

The proof of this is out of the scope of this paper and involes Lie Algebra so we will be omitting it.

Lemma 5.5. *If X and Y are manifolds and $f, g : X \to Y$ are homotopic smooth maps. For every p, the induced cohomology maps $f^*, g^* : H^p(Y) \to H^p(X)$ are equal.*

Proof. By the Lemma 5.4, the maps i_0^* and i_1^* are equal. Since f and g are homotopic smooth maps, they are smoothly homotopic so there is a smooth function $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. This means that $f = H \circ i_0$ and $g = H \circ i_1$ so by Proposition 5.3,

$$f^* = (H \circ i_0)^* = i_0^* \circ H^* = i_1^* \circ H^* = (H \circ i_1)^* = g^*.$$

■

Now we are ready to prove the main theorem of the section.

Theorem 5.6 (Homotopy Invarience)**.** *If X and Y are homotopy equivalent manifolds, then their pth cohomology groups are isomorphic for every p i.e. $H^p(X) \cong H^p(Y)$.*

Proof. Let $f : X \to Y$ be a homotopy equivalence with homotopy inverse $g : Y \to X$. By the Whitney approximation theorem, there exists smooth maps $\tilde{f} : X \to Y$ homotopic to f and $\tilde{g} : Y \to X$ homotopic to g. This means that $\tilde{f} \circ \tilde{g} \simeq f \circ g \simeq \mathrm{Id}_Y$. By Lemma 5.5 and Proposition 5.3, we have

$$(\tilde{f} \circ \tilde{g})^* = (\mathrm{Id}_Y)^* \implies \tilde{g}^* \circ \tilde{f}^* = \mathrm{Id}_{H^p(Y)}$$

Similarly, we have $\tilde{g} \circ \tilde{f} \simeq g \circ f \simeq \mathrm{Id}_X$ so $\tilde{f}^* \circ \tilde{g}^* = \mathrm{Id}_{H^p(X)}$. Therefore, the map $\tilde{f}^* : H^p(Y) \to H^p(X)$ is an isomorphism. ■

Because all homeomorphisms are homotopy equivalences, we get the following result:

Corollary 5.7 (Topological Invarience)**.** *If X and Y are homeomorphic manifolds, then their cohomology groups are isomorphic i.e. the cohomology groups are topologically invarient.*

This is truly an amazing result since this is something we wouldn't expect based on the definition of cohomology groups. De Rham cohomology groups were defined on smooth structures on manifolds so we have no reason to expect that differentiable smooth structures on topologically equivalent manifolds produce the same cohomoloy groups.

Now recall that when we introduced the De Rham cohomology groups, it was to investigate when closed forms are exact since this is not necessarily the case. This is what the Poincaré lemma tells us:

Theorem 5.8 (Poincaré Lemma)**.** *If U is a star-shaped open subset of $\mathbb{R}^n$, then $H^p(X) = 0$, or the trivial group, for $p \geq 1$.*

Recall that a star-shaped set U is a set where there exists a $c \in U$ such that for every $x \in U$, the line segment between c and x is contained in U. Before we go on to the proof, one of the most important consequences is this:

Corollary 5.9. *Every closed form on X is locally exact i.e. each point in X has a neighborhood on which every closed form is exact.*

Proof. Every point on X has a neighborhood that is diffeomorphic to an open ball in $\mathbb{R}^n$. Since $\mathbb{R}^n$ is star-shaped and cohomology groups are diffeomorphically invarient, by the Poincaré Lemma or Theorem 5.8, we are done. ■

Now we prove the Poincaré Lemma.

Lemma 5.10. *If X is a contractible manifold, then $H^p(X) = 0$, or the trivial group, for $p \geq 1$.*

The idea behind contractibe manifolds is that we can squeeze it until it becomes a point. The cohomology group for a point is clearly the trivial group so we use homotopy invarience to prove the rest of this lemma.

Proof. Because X is contractible, there exists an x such that the constant map $c_x : X \to X$, which sends all of X to x, is homotopic to the identity map. Now let $\iota_x : \{x\} \to X$ be the inclusion map. This means that $c_x \circ \iota_x = \mathrm{Id}_{\{x\}}$ and $\iota_x \circ c_x \simeq \mathrm{Id}_X$ so ι_x is a homotopy equivalence. By Theorem 5.6, we have $H^p(X) = H^p(\{x\}) = 0$ since $\{x\}$ is a 0-manifold. ■

Proof of the Poincaré Lemma. A star-shaped domain is contractible because of the straight-line homotopy:

$$H(x, t) = c + t(x - c)$$

so the constant and identity map are homotopic. By Lemma 5.10, we have $H^p(U) = 0$. ■

References

[GP10] V. Guillemin and A. Pollack. *Differential Topology.* AMS Chelsea Publishing. AMS Chelsea Pub., 2010.

[Gre09] Patrick Greene. De rham cohomology, connections, and characteristic classes. 2009.

[Lee12] John M Lee. *Introduction to Smooth Manifolds.* Graduate Texts in Mathematics. Springer New York, New York NY, 2012.

[McN14] Redmond McNamara. Introduction to de rham cohomology. 2014.